The Latest Frege*

1. Opening

Many authors believe that the manuscripts of Frege from 1924–5 are not of high quality. They are rather a product of his emotional despair and theoretical dead-end which he reached in the last years of his life. Such is also the judgement of Michael Dummett delivered in his seminal book *Frege: Philosophy of Language*. According to Dummett, 'the few fragmentary writings of Frege's final period—1919–1925—are not of high quality: they are interesting chiefly as showing that Frege did, at least at the very end of his life, acknowledge the failure of the logicist programme' (Dummett 1981, p. 664).

In this paper we will try to show that the widely accepted negative assessment of Frege's latest writings is due to a lack of understanding of their true idea. In fact, the change in Frege's mind in the last two or three years of his life was a result of long considerations on a severe tension in his founding intuitions. The change itself made his logic more coherent and, thus, is of utmost theoretical importance.

2. Frege's Intuition-Dualism and its Elimination

The attentive reader of Frege can easily find a striking dualism in (among other things) his attitude towards the role of intuition in mathematics. On the one hand, his aim was to construct a deductive system of inference free from the gaps of intuition. He often repeated that 'one may not appeal to intuition as a means of proof'.

On the other hand, Frege concedes (on the same page) that 'it is permissible to use intuition as a helpful expedient in pinning down [*Festhaltung*] an idea'

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(1881a, 32 n.). In fact, the whole project for a concept-script—which must demonstrate also how the logical proof is done—was built on the latter assumption. Frege's aim was to suggest a *graphically* perfect language, the purpose of which was to *show* how things went in logic.[1]

From here it follows a deflationary understanding of logic. Apparently, Frege was convinced that 'if our language was logically more perfect, we would perhaps have no further need of logic, or we might read it off [*ablesen*] from the language' (1915, p. 252). In the *Tractatus*, Wittgenstein transformed this conditional into a strict deflationary *theory* of logic: 'We can actually do without logical propositions; for in a suitable notation we can in fact recognise the formal properties of propositions by mere inspection of the propositions themselves' (Wittgenstein 1922, 6.122).

This be as it can, one point is certain. Frege's concept-script presupposes a geometrical imagination which must help in grasping logical sense. In this connection, some authors have suggested that Frege's concept-script conveys a 'perceptual model of understanding'. Indeed, 'Frege's staunch semantical realism requires that the nature of human "understanding" be interpreted in a manner quite analogous with the interpretation of "seeing" on the (non-intuitive) realist model of visual perception' (Schweizer 1991, p. 264). The very use of the concept of grasping [*fassen*] suggests that there is something solid—a figure, *Gestalt*—which is to be taken hold of and further picked up.

In this paper we will accept that Frege's controversial statements on intuition are a result of the fact that he was concerned with two different types of intuition. On the one hand, he was convinced that by thinking, there is no place for sensual intuition. On the other hand, the logical inference, the judgement, the deduction

[1] This project is not to be confused with the project for a *perfect*ly grounded *language*. According to the latter project, 'in science the purpose of a proper name is to designate an object determinately; if this purpose is unfulfilled, the proper name has no justification in science' (1906, 178). Roughly speaking, whereas Wittgenstein developed the former project, Russell developed the latter.

are made when we recognise intellectually—via intellectual intuition—some geometrically organised structures.

3. The Need for New Symbolism

Frege's theoretical reason for the project of concept-script was that 'our attention is directed by nature to the outside'—to senses. So we think with necessity in symbols. At least by humans, a 'concept is first gained by symbolising it; for since it is, in itself, *imperceptible*, it requires a perceptible representative in order to appear to us' (1881b, p. 84). Here it is important to notice that this problem does not concern language only, as Dummett assumes. Thought must be expressed in proper *symbolism*, and language is only one part of this symbolism. Incidentally, Dummett was not the first misled on this point. Wittgenstein, who almost literally repeated Frege on the need to 'perceptually represent' thinking, also limited the project for perfect symbolism to language only.

Frege's world of logic—the world of deduction—is objective but not spatial. Amongst its denizens are the numbers. 'Spatial predicates are not applicable to them' (1884, § 61). The same is true of concepts, of truth and falsehood and, of course, also of logical forms. Now the problem is that this non-spatial world is—at least by humans—of necessity to be grasped intuitively. Usually, we make this by the way of language. The language, however, is not invented in order to accomplish scientific, exact purposes. That is why we must invent and introduce an appropriate—felicitous for our intuition—symbolism.

Frege saw this task as his main priority. His motive was the conviction that 'without the great invention of symbols which call to mind that which is absent, invisible, . . . the course of our ideas cannot gain its freedom from this: it would still be limited to that which our hand can fashion, our voice intone' (1881b, p. 83). On the contrary, 'if we produce the symbol of an idea which a perception has called to mind, we create in this way a firm, new focus about which idea gather' (ibid., pp. 83–4).

It remained unclear, however, what the theoretical ground of Frege was to assume that an intuitive representation of the non-spatial world of logic is possible at all. How exactly do symbols with certain spatial characteristics, call up in our mind something invisible? Is this a process of decoding? If yes, according to which rules is this deciding made? Even more puzzling is Frege's assumption that precisely the *geometrical order* of logical signs should demonstrate the *logical order* of logical objects.

In what follows we will accept that, in real fact, Frege had no grounds for this presupposition. Realising this, in the last two–three years of his life he assumed that logical signs and objects are of the same order—of geometrical order.

4. The Geometrical Character of the New Symbolism

Frege's idea was that the concept-script should serve for 'perspicuous [*anschauliche*] representation of the forms of thought' (ibid., p. 89). It should be nothing but an *optical* instrument with which we could grasp the logical forms 'without much ado'. These very forms are nothing but shapes [*Gestalten*] which can be 'generally sharply defined and clearly distinguished' (ibid., p. 87).

Thus the perspicuity of symbolism is to be achieved through the spatial relations of the symbols. Frege was convinced that 'the spatial relations of written symbols on a two-dimensional writing surface can be employed in far more diverse ways to express inner relationship than the mere following and preceding in one-dimensional time. . . . In fact, simple sentential ordering in no way corresponds to the diversity of logical relations through which thoughts are interconnected.' (ibidem.)

Actually, the whole concept-script of Frege is built by geometrical and means.

(i) The sign for assertion consists of a content stroke and judgement stroke. Since the content stroke denotes the one-levelled combination (interweaving) of ideas,[2] it is symbolised by a *horizontal* stroke. The act of assertion, which intro-

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[2] Incidentally, the interweaving of ideas in judgement, as well as of objects in states of affairs

duced into judgement another dimension—that of the will—is symbolised by a *vertical* stroke (Frege 1879, § 2).

(ii) Two propositions are connected by conditional stroke [*Bedingungsstrich*]. Their alternative value here is symbolised via posing two content strokes one *above* the other. The conditional stroke relates these opposite contents.

(iii) The generality is expressed by way of *concavity* which comes to symbolise that it treats singular terms intensionaly. This way of symbolising accepts that the logical constants in logic *show* themselves; they cannot be expressed.

(iv) But even the most elementary concepts in Frege's logic are recognised via their geometrical properties. Thus Frege assumed that 'where logic is concerned, it seems that every combination of parts results from completing something that is in need of supplementation [*Ergänzungsbedürftigkeit*]' (1919, p. 254). A typical example here is the concept of function. Now the 'need of supplementation' by functions is denoted by clearly intuitive means: it is expressed by the *space* within the brackets in *f* (   ) (see 1924, pp. 271–2).

Here it is obvious that all the four most basic concepts/operations of Frege's logic are *ineffable*.

5. Frege's *Begriffsschrift* as Ideography

Thus Frege's program for conceptual notation is nothing but a conventional *ideography*—a means for a graphical representation of ideas. On this point Frege was correctly understood by his followers. So Peano projected a 'construction of graphic symbolism, or ideography, capable of representing all the ideas of logic' (Peano 1973, p. 190). This symbolism—Peano insisted—is not merely a tachygraphy. Russell also accepted a program for authentic graphical representation of logic. Indeed, *Principia Mathematica* offers 'a symbolism especially designed to represent the ideas and processes of deduction which occur in' it. It 'aids the in-

(*Sachverhalt*) is expressed by Russell graphically with the ' ' sign in the blue-print to the famous 'On Denoting'—'On Fundamentals' (Russell 1994), and also in a paper published shortly afterwards: 'The Theory of Implication' (Russell 1906).

tuition in regions too abstract for the imagination readily to present to the mind the true relation between the ideas employed' (Russell and Whitehead 1910, p. 2).

That Frege's *Begriffsschrift* is an ideography—not a revolutionary new means for extra-spatial reasoning but rather a dimensional (spatial) representation of ideas—is shown (i) by the etymology of this concept. Frege, namely, borrowed it from Trendelenburg, and Trendelenburg knew it from Carl Hindenburg's 'combinatorial school'.[3] The latter, in turn, translated *Begriffsschrift* from the French *ideographie*, which was developed near the end of the eighteenth-century from D'Alambert and Condillac.

(ii) This can also be seen in Frege's conviction that *Begriffsschrift* is not something different but merely a further development of the symbolism already accepted in science and mathematics. The signs in mathematics, for example, express to a great extent contents. 'What [they] still lack is the logical cement [*Mörtel*] that will bind these building stones firmly together' (1881a, p. 13). The cement (logic) and the bricks (scientific/mathematical theories), however, lie in one world—the world of geometry.

Much of the success of such a program depends on how the symbols of the perfect language are devised. It was precisely on this point that Frege failed. His specific suggestion—the baroque notation of *Begriffsschrift* and *Grundgesetze*—failed to persuade the public. The new form of symbolism was not accepted by the logicians—even by those who were most sympathetic with his logical theories. Russell, for example, preferred Peano's notation, highly appreciating it precisely for its advantages in representing.

Wittgenstein was perhaps the only logician who tried to develop Frege's symbolism further. (This point can be scarcely a surprise: Wittgenstein was the only real *pupil* of Frege in logic.) This is clearly seen in *Tractatus* 6.1203, where he suggested an 'intuitive method' for recognising an expression as tautology. Unfortunately, it was not adopted by his numerous admirers. Nevertheless, the later

[3] See, for example, Hindenburg 1803.

Wittgenstein did not reject it. He used it with confidence, for example, in his lectures of 1935 (see Wittgenstein 1979a, p. 136). This shows the theoretical importance of that undertaking.

6. Frege's Geometrical Turn in 1923

It was this implicit geometrism of Frege's logic that urged him to accept the geometrical foundation of mathematics in 1923, and more precisely, the geometrical nature of mathematical objects.

We must bear in mind here that Frege realised the geometrical origin of mathematics (and philosophy) only after long deliberations. Only towards the end of his days he saw that the 'infinite in the genuine and strictest sense of the word' follows from the geometrical source of knowledge (1924, p. 273).

Perhaps what made Frege realise the intuitive character of logic was the stress laid on the geometrical character of logical symbolising in Frege's own logic by Wittgenstein during his three visits to Jena. Here one must be reminded of Ruben Goodstein's report that Wittgenstein once told him how by his first discussion on logic with Frege the latter 'wiped the floor with him. Wittgenstein returned to England very disheartened, but a year later he [Wittgenstein] sought another interview with Frege and this time "he wiped the floor with Frege"' (Goodstein 1972, p. 272). Now Peter Geach, who was told only the first part of the anecdote, is confident (see Anscombe and Geach 1961, p. 130) that its second part is 'spurious' (see Geach 1988, xiv). If we have in mind Geach's pro-Fregean biases, this assessment is not a surprise. Against it it can be pointed out that, as matter of fact, Wittgenstein communicated the story to Goodstein more than 12 years earlier (in 1931–5) than to Geach (1945–7). It is reasonable to expect that Goldstein's story, which were delivered much earlier than this of Geach, is more authentic.

Peter Hacker believes that the decisive turn in the Frege–Wittgenstein discussion came after the second visit of Wittgenstein (see Hacker 1996, p. 307). A reason for this is his letter to Russell from 26 December 1912, which reads: 'I had a long discussion with Frege about our Theory of Symbolism [sic!] of which, I think, he roughly understood the general outline. He said he would think the matter over' (Wittgenstein 1974, p. 17).

We have good grounds to suppose, however, that Wittgenstein succeeded in *intriguing* Frege (if not in his ultimate persuading) in the inconsistency of his logical theory only by his third visit to him in December 1913. We can guess the content of their discussion from the fact that immediately after they met in Jena, Wittgenstein formulated the doctrine of 'logical showing' (in 'Notes Dictated to G. E. Moore') thus: 'In "aRb", "R" is not a symbol, but [the geometrical fact] that "R" is between one name and another symbolises' (Wittgenstein 1979b, p. 109). This was the main innovation in the 'Notes Dictated to G. E. Moore', which can not be found in 'Notes on Logic' dictated in September 1913.

Frege, who always expressed to his pupil (for example, on 16 September 1919) the hope 'to find something by you what makes complete what I already found. . . to learn to see with your eyes' (Frege 1989, p. 21), was much slower indeed in catching the point of this lesson. Only after 1923 (after he received Wittgenstein's *Tractatus* just printed!) did he find the courage to radically change his philosophy of mathematics in accordance with Wittgenstein's remarks.

Realising that his concept-script has a geometrical character, Frege now accepted that the (mathematical) objects they demonstrate ('show') are geometrical. Only now was Frege's logic free from its fatal dualism. Its main point became the expressed assumption that both philosophical and mathematical knowledge have geometrical sources. The mathematical deduction is based on intuition.

7. Logical Showing

An implicit advantage of the understanding expressed in this paper is that it suggests a new, perspicuous treatment of the, otherwise, enigmatic theory of 'logical showing' accepted by both Frege and Wittgenstein.

Now in the *Tractatus*, Wittgenstein often insists that the logical properties of symbols ‘show themselves’; that they are ineffable. More than twenty years ago Peter Geach demonstrated that there are also ineffable points in Frege’s logic (see Geach 1976).

The interpreters of Frege and Wittgenstein find the theory of ‘logical showing’ paradoxical; a ‘dialectical matter’ (Gottfried Gabriel). At that, they face insurmountable difficulties by trying to specify where *exactly* it is valid. Thus Geach assumes that it comes to light simply ‘when we reflect upon logic’ (Geach 1976, p. 56), and Gabriel—by making ‘categorical differences’.

In contrast, according to our understanding, both Frege and Wittgenstein accepted that what cannot be said in logic, but is *shown* in it, are just its main concepts: judgement, function, logical constants. So, if we are good enough in fixing the shape [*Gestalt*] of thinking—the *perfect symbolism*—then the whole discipline of logic would become superfluous. The perfect logical symbolism suggested will not be merely ‘a “wink” with the help of “picturesque expressions”’, as some authors suggest (see Gabriel 1991, p. 84), but a thoroughly ‘perspicuous *representation*’ of human thought.

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