

# Algebra of Logic, Quantification Theory, and the Square of Opposition\*

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## 1 Introduction: Logic as Calculus and Logic as Language

1967 was one of the most important years for the historiography of modern logic. In that year Jean van Heijenoort published his *From Frege to Gödel*, according to its subtitle “A Source Book in Mathematical Logic, 1879–1931”. The seminal character of this collection is due to the fact that for the first time basic texts from the history of mathematical logic were gathered together, most of them for the first time in the English language. Its formative influence on the historiography of logic had, however, some negative effects. In concentrating on the logical tradition starting with Frege’s *Begriffsschrift* (Frege 1879) it largely ignored the logical direction with which the 19th century rebirth of the interest in formal logic started: the algebra of logic. In saying that “the second half of the nineteenth century saw a rebirth of logic” van Heijenoort was clearly aware of the pre-Fregean algebraic tradition of logic (vi). However, he sees this tradition suffering from a number of limitations (*ibid.*):

It tried to copy mathematics too closely, and often artificially. The multiplicity of interpretations of what became known as Boolean algebra created confusion and for a time was a hindrance rather than an advantage.

This led van Heijenoort to his final assessment that the period of the algebra of logic “would, no doubt, leave its mark upon the history of logic, but it would not count as a great epoch” (*ibid.*).

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According to van Heijenoort “the great epoch in the history of logic did open in 1879, when Gottlob Frege’s *Begriffsschrift* was published” (*ibid.*). The reasons were, according to his opinion, that the *Begriffsschrift* freed logic from the artificial connection with mathematics, at the same time preparing “a deeper interrelation between these two sciences”. Among his criteria van Heijenoort stated that “it presented to the world, in full-fledged form, the propositional calculus and quantification theory” (*ibid.*). It seems to be obvious that this should imply the assertion that there was no quantification theory in the algebra of logic.

I feel first of all urged to a short comment on van Heijenoort’s assertion that in the algebra of logic the connection between mathematics and logic was too narrow. Even if he were right, this insight hides the really interesting point, the fact namely that the development of modern structural mathematics and modern logic ran basically parallel. Most algebraists of logic were also involved in the creation of structural mathematics and the mathematics involved in logic was in fact this new mathematics (cf. Peckhaus 1997, chs. 5 and 6). But now back to quantification theory and van Heijenoort!

It was in the same year 1967 that van Heijenoort published his paper “Logic as Calculus and Logic as Language” (1967b) which was even more important for our today’s standard understanding of the early directions in modern logic than his source book. Nevertheless, despite its significance, I insist on my judgement that it is a model for sloppiness, one-sidedness, and ignorance, showing a dreadful insensibility for the complexity of the historical development.

Van Heijenoort took up Leibniz’s distinction between the “calculus ratiocinator” and the “lingua characteristica” both of which provide tools for Leibniz’s *scientia generalis*. But he interpreted this distinction as standing for two kinds of approaching logic. The dichotomy between logic as calculus and logic as language is furthermore used to distinguish between the two big traditions in the newer history of logic, the algebra of logic on the one hand and the tradition opened with Gottlob Frege’s *Begriffsschrift* of 1879, the latter sometimes identified with mathematical logic. In doing so, van Heijenoort opposes George Boole and Leopold Löwenheim on the algebraic side with Gottlob Frege and Bertrand Russell on the side of mathematical logic. He mentions Ernst Schröder, the most important representative of the German algebra of logic, and Charles S. Peirce, his ingenious American colleague and inspirer, only in passing, but in beating Boole and Löwenheim, he also hits Schröder and Peirce, of course.

Frege’s logic is described as a universal language which does not have, because of its universality, any separate semantics. “The universal formal language supplants the natural language”, he writes (327),

and to preserve, outside the system a notion of validity based on intuitive set theory, does not seem to fit into the scientific reconstruction of the language.

The algebraic propositional calculus uses, on the contrary, a model-theoretic approach. Algebraic systems always need interpretations of operation signs and categories, i. e., classes of concepts. They can clearly be divided into a structural, or syntactic, and a semantical side, the latter providing the interpretations of the figures used.

Van Heijenoort maintained that the universality of Frege's *lingua characteristic* is founded in Frege's quantification theory, a feature lacking in Boole's propositional calculus (324–325). This is in fact true, but van Heijenoort withholds some other verities. He does not mention that almost at the same time when Frege published his quantification theory, an equivalent one was developed in the school of Charles S. Peirce, later taken over by Schröder, and extensively discussed in the second volume of the *Vorlesungen über die Algebra of Logik* (1891). Therefore it makes no sense to use quantification theory to distinguish between the algebra of logic and the Frege kind of mathematical logic.

Nevertheless, the story is quite complicated, and it is the aim of my lecture today, to shed some light on it. In the meantime the historiography of logic (but not the philosophy of logic and mathematics) has overcome van Heijenoort's one-sidedness. Already van Heijenoort's doctoral student and biographer Irving H. Anellis wrote that the "distinction between the 'algebraic' and the 'quantification-theoretic' traditions is artificial at best" (1994, 111). Anellis thought that van Heijenoort's distinction was mainly due to Bertrand Russell's influence (*ibid.*, 103–104; cf. Jean-Yves Béziau's review of Anellis' book Béziau 1998/2000). Among recent work I would like only to mention Geraldine Brady's book *From Peirce to Skolem* (2000) which focusses on the Peirce-Schröder tradition in the development of first-order logic, and Ivor Grattan-Guinness's seminal survey *The Search for Mathematical Roots* (2000) which gives deep insights into the interrelations between the different traditions in logic.

In the following I will reconsider the differences in quantification theory between the two traditions by taking a common point of reference into account, their positions toward the Aristotelian-Scholastic theory of judgement.

## 2 The Square of opposition

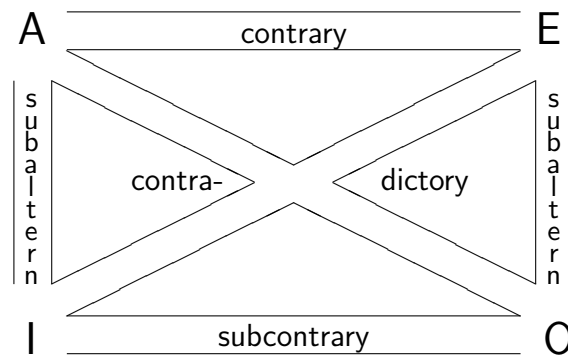
### 2.1 The Aristotelian-Scholastic Model

The Aristotelian-Scholastic theory of judgement is based on the a special kind of propositions, the so-called "standard forms" which contains some sort of

quasi-quantification. The list of standard forms runs as follows:

All $S$ are $P$	$SaP$
No $S$ is $P$	$SeP$
Some $S$ are $P$	$SiP$
Some $S$ are not $P$	$SoP$

Such standardized judgements consist of two concepts, the subject  $S$  and the predicate  $P$ , which are combined with the help of a connector, the copula, expressed by the words “is” or “are”. Two of these judgements are universal, indicated by the affirmative quantifier “all” and the negative quantifier “no”, the two other judgements are particular, particular-affirmative and particular-negative. The respective quantifier “some” as well as the universal quantifiers usually refer only to the subject term. The relations between these judgements can be illustrated with the help of the square of opposition:

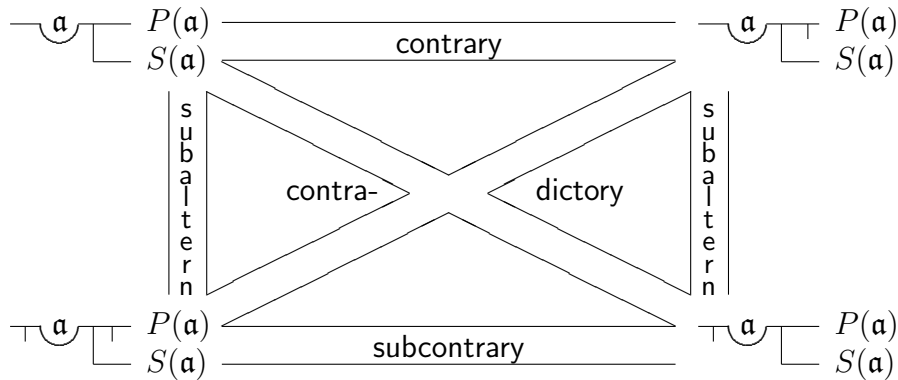


## 2.2 Frege’s Interpretation

The modern interpretation of the standard forms is the following:

All $S$ are $P$	$SaP$	$\bigwedge_x(Sx \rightarrow Px)$
No $S$ is $P$	$SeP$	$\bigwedge_x(Sx \rightarrow \neg Px)$
Some $S$ are $P$	$SiP$	$\neg \bigwedge_x(Sx \rightarrow \neg Px)$ , or $\bigvee_x(Sx \wedge Px)$
Some $S$ are not $P$	$SoP$	$\neg \bigwedge_x(Sx \rightarrow Px)$ , or $\bigvee_x(Sx \wedge \neg Px)$

This interpretation goes back to Frege who used a similar one in his *Begriffsschrift*. There one can find the following square which is here reproduced with a slight modification. Frege mistakenly described the relation between I judgement and O judgement as “contrary” instead of “subcontrary” (Frege 1879, 24).



In Frege's logic only negation, implication and universal quantification are used. The quantifiers range over a domain of individuals marked by a Gothic letter written into a "Höhlung", a concavity. About its necessity Frege writes that "it delimits the scope of the generality signified by the letter. Only within its scope does a Gothic letter retain its meaning" (Frege 1879, 20, transl. Beaney, ed., 1997, 71). In signifying the range of a quantifier Frege made the decisive step towards modern quantification theory. This step was made possible by Frege's revision of the understanding of a judgement. It is no longer the connection of the concepts of or the terms for subject and predicate with the help of a copula, but a function of arguments. The judgement " $A$  is  $\Phi$ ", or, more exactly " $A$  has the property  $\Phi$ ", is therefore expressed by  $\vdash \Phi(A)$  (cf. Frege 1879, 18). A side-effect of this new understanding is that Frege's notation isolated the statements used in a complex proposition from the logical structure of this proposition. Frege's explanations given in a letter to Ludwig Darmstaedter of July 1919 are instructive in this respect (Frege 1983, 273, transl. taken from Beaney, ed., 1997, 362):

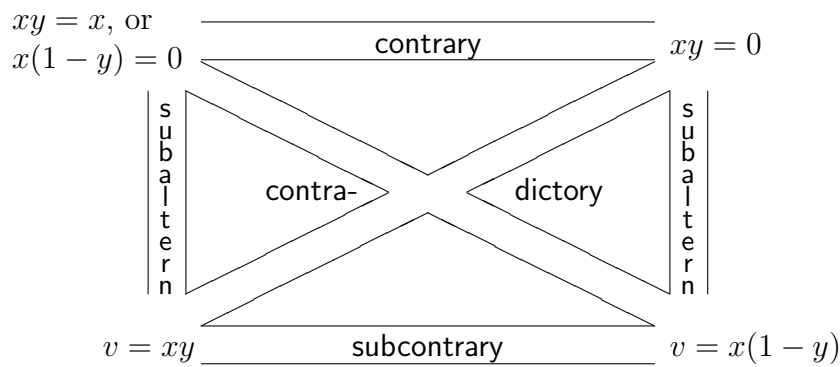
So I do not begin with concepts and put them together to form a thought or judgement; I come by the parts of a thought by analyzing the thought. This marks off my *Begriffsschrift* from the similar inventions of Leibniz and his successors, despite what the name [*Begriffsschrift*] suggests; perhaps it was not a very happy choice on my part.

### 2.3 Boole's Interpretation

Contrary to Frege, the algebraists initially stuck to the traditional concept of the judgement as combination of concepts. In *The Mathematical Analysis of Logic* which can be regarded as the birth document of modern mathematical logic, in the sense that this kind of logic enjoyed a continuous development subsequently, Boole mentioned that his juvenile interest for logic had been revived by the priority struggle between the Scottish philosopher Sir William

Hamilton and the London mathematician and philosopher Augustus De Morgan about the quantification of the predicate (cf. Boole 1847, 1). This struggle concerned a modification of the traditional theory of the judgement, and although neither Hamilton nor De Morgan had any priority in this respect, it nevertheless indicated a new interest in logical research in Great Britain at that time (cf. Heath 1966).

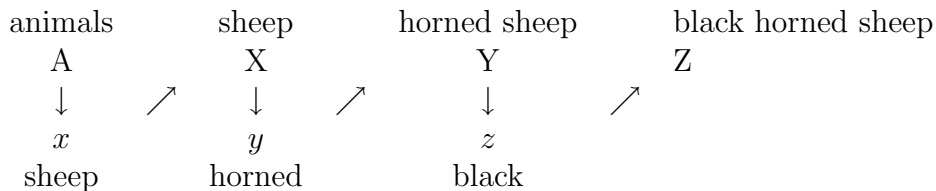
In respect to the topics treated Boole followed closely the model of traditional syllogistic. Using Boole's symbolism the square of opposition would look like this:



In Boole's notation, capital letters stand for all members of a certain class. The small letters are introduced as follows (Boole 1847, 15):

The symbol  $x$  operating upon any subject comprehending individuals or classes, shall be supposed to select from that subject all the  $X$ s which it contains. In like manner the symbol  $y$ , operating upon any subject, shall be supposed to select from it all individuals of the class  $Y$  which are comprised in it and so on.

Take  $A$  as the class of animals, then  $x$  might signify the selection of all sheep from these animals, which then can be regarded as a new class from which we select further objects, etc. This might be illustrated by the following example:



What we have done here is a successive selection which leads to individuals being common to the classes  $A$ ,  $X$ ,  $Y$ , and  $Z$ .  $xyz$  stands for animals that are sheep, that are horned and black. It can be regarded as the logical product of some common marks or common aspects relevant for our selection. In his major work *An Investigation of the Laws of Thought* (1854) Boole gave

up the distinction between capital and small letters, thereby significantly simplifying his notation.

The symbol 1 denotes the universe, and therefore it is consequent that if the class X is determined by the symbol  $x$ , the class not-X has to be denoted by the symbol  $1 - x$ .  $1 - x$  forms the supplement to  $x$ , thus  $x(1 - x) = 0$ , where 0 symbolizes nothing or the empty class. Now we can consider Boole's interpretation of the universal-affirmative judgement. The universal-affirmative judgement "All Xs are Ys" is expressed by the equation  $xy = x$  or, by simple arithmetical transformation,  $x(1 - y) = 0$  (Boole 1847, 22):

As all the Xs which exist are found in the class Y, it is obvious that to select out of the Universe all Ys, and from these to select all Xs, is the same as to select at once from the Universe all Xs.

The universal-negative judgement "No Xs are Ys" asserts that there are no terms common to the classes X and Y. All individuals common would be represented by  $xy$ , but this product forms the empty class. The particular-affirmative judgement "Some Xs are Ys" says that there are some terms common to both classes forming the class V. These terms are expressed by the elective symbol  $v$ . The judgement is thus represented by  $v = xy$ . Boole furthermore considers to use  $vx = vy$  with  $vx$  for "some X" and  $vy$  for "some Y", but observes "that this system does not express quite so much as the single equation [...]" (p. 22–3). The particular-negative judgement "Some Xs are not Ys" can be reached by simply replacing  $y$  in the last formula with  $1 - y$ .

As mentioned, Boole revised his logical theory in his later logical work, especially in his seminal *An Investigation of the Laws of Thought* (Boole 1854). Although he developed there a theory of logical functions of which first ideas can already be found in the earlier work, and although he introduced "secondary propositions" as propositions about propositions, thereby having necessary elements of the Fregean paradigm of logic at hand, no trace of a quantification theory can be found. So van Heijenoort seems to be right in denying the existence of quantification theory in the algebra of logic.

But Boole does not represent all of the algebra of logic, and so he should not be taken for all.

### 3 Schröder's Quantification Theory

In the following I will concentrate on the German algebraist of logic Ernst Schröder who started to adopt and revise Boole's algebra of logic 20 years after Boole. He published the first fruits of his research in his *Der Operationskreis des Logikkalküls* (1877).

### 3.1 Misunderstanding Frege's Quantification Theory

Schröder's *Operationskreis* gives further evidence for the assumption that it was the algebraic paradigm of regarding judgements as combinations of concepts that prevented the early algebraists of logic to interpret the old standard forms of the syllogism in terms of modern quantification theory. Notorious in this respect is Schröder's review of Frege's *Begriffsschrift*, published in 1880 (Schröder 1880). After Hans Sluga's influential assessment this review is infamous for its hidden polemics which was said to have been to a great extent responsible for Frege's ineffectiveness in his time: "It tore the book apart—in the politest possible way. There was a good deal of hurt vanity in Schröder's words" (Sluga 1980, 68). A close reading of this review suggests, however, to ascribe Sluga's assessment to the logical folklore (cf. Vilkko 1998). The review is polemical in parts, no doubt, but this style of argument may be ignored as typical for the time in which it was written. If this is done one becomes aware that the reviewer took a favorable view on Frege's book, using the advantage, however, to advertise the new subject symbolic logic as such. This made it possible to direct the interest of the readers not only on Frege's logic, but also on the algebraic direction started by Boole. So the review was some sort of comparative advertising, and it is clear that Schröder, as an algebraist, had to come to the conclusion that the algebra of logic was the better realization of common goals.

For our topic the review is of special interest because Schröder directly referred to Frege's quantification theory. He admitted some shortcomings in Boole's treatment of particular judgements which found in Boole's theory "only an inadequate, in a rigorous reading, no expression", as he said (Schröder 1880, 91). This was harsh, but it was harsh against Boole, not against Frege. Schröder justified his assessment by the following argument (*ibid.*, 90f.):

The indefinite factor  $v$ , which is used by Boole to express in the first part of the logical calculus the statement "some  $a$  are  $b$ " in the form of  $va = vb$ , does not serve its purpose because this equation is always identically fulfilled by the assumption  $v = ab$ , even in the case that no  $a$  is  $b$ . In the section on "universality" Frege justly gives such stipulations which allow him to express these judgements indubitably. I will not follow him slavishly in this respect, rather show that they do not justify his further deviations from Boole's notation, but that the latter can also be modified and extended by analogy. Author reaches this essentially in the way that he introduced Gothic letters in the meaning of general signs and stipulated a notation to negate this universality [...].

This is not true! Frege's syntactical sign for universality is the concavity. The



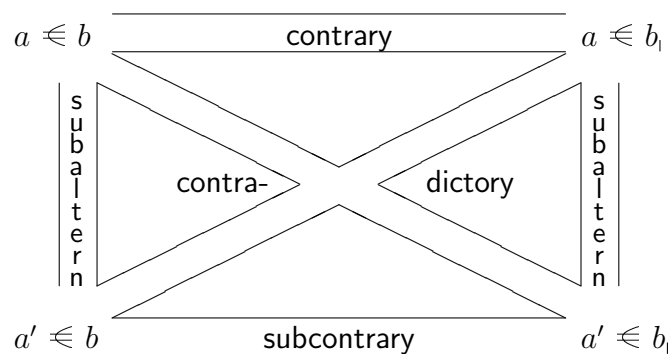
Gothic letter signifies the scope of the quantifier, i. e., the range of arguments which can be used in the quantified formula. This quote shows that Schröder (at least in 1880) simply did not grasp the concept of the scope of a quantifier. His easy modification of the Boolean notation consisted in plainly restricting the formula  $va = vb$  by introducing the sign  $\neq$  for “not equal” and stipulating that  $va \neq 0$  or  $ab \neq 0$  which together would also express that some  $a$  are  $b$  (*ibid.*).

### 3.2 Schröder’s Interpretation of the Judgements

The review was published in 1880. It belongs to Schröder’s early period of logical research. His mature theory is represented by the three volumes of his *Vorlesungen über die Algebra der Logik* with which he brought the algebra of logic to its provisional perfection. This seminal and influential work was published between 1890 and 1905 (Schröder 1890–1905). It remained, despite of its 2155 pages, unfinished.

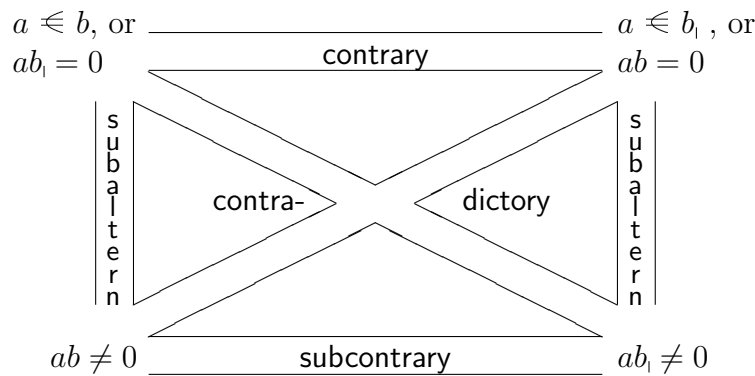
The first volume was devoted to the calculus of domains (arbitrary elements separate from each other) and the calculus of classes. The second volume in two parts dealt with the calculus of propositions, and the final third one brought the “Algebra and Logic of Relatives”, finished only in its first, algebraic, part. The algebra and logic of relatives was regarded by Schröder as the central tool for providing an algebraic scientific universal language (cf. Peckhaus 1991; 1994; 1997, ch. 6).

The second volume contains among other topics considerations on “Die Syllogismen der Alten”, i. e., “The Syllogisms of the Ancient” (§ 42, 221). There Schröder gave a very simple interpretation of the standard forms which can be summarized in the following square of opposition:



Small letters stand for universal classes, the prime indicates particularity, negation is expressed by a postponed stroke, the copula is represented by Schröder’s central operator, the subsumtion, representing in the class calculus the relation “being sorted into ... or being equal to ...”.

Earlier in the book, in a lecture dealing with the traditional division of categorical judgements in respect to quality and quantity, Schröder brings a more sophisticated discussion of the standard forms taking up his earlier criticism of Boole's treatment of particular judgements (§ 33, 90–93). He again argues that the Boolean calculus is incapable of representing particular judgements. Schröder's solution, similar to that given in 1880, comes along, however, as an expedient, for he again introduces a new operator,  $\neq$  for “not equal”. Schröder's modified square of opposition looks like this:



Again, like in Boole's work, we find an interpretation of the standard forms in terms of concepts. No trace of a quantification theory can be found, at least in this context.

But the same volume nevertheless disproves the assessment implied by van Heijenoort that there is no quantification theory at all in the algebra of logic. Schröder's quantification theory is, however, not connected to the theory of the syllogism.

The step from the calculus of classes to the calculus of propositions is taken with the help of an alteration of the basic interpretation of the formula used. Whereas the calculus of classes was bound to a spatial interpretation especially in terms of the part–whole relation, Schröder now employed a temporal interpretation. He took this idea from Boole's *Laws of Thought* (1854, 164–165). This may be illustrated by considering subsumption as the basic connecting relation. In the calculus of classes  $a \in b$  means that the class  $a$  is part of or equal to the class  $b$ . In the calculus of propositions this formula may be interpreted in the following way (Schröder 1891, § 28, p. 13):

The time during which  $a$  is true is completely contained in the time during which  $b$  is true, i. e., *whenever* [...]  $a$  is valid,  $b$  is valid as well. In short, we will often say: “If  $a$  is valid, then  $b$  is valid,” “ $a$  entails  $b$ ” [...], “from  $a$  follows  $b$ .”

Related to the general calculus of domains, Schröder then introduces two new logical symbols, the “sign of products”  $\prod$ , and the “sign of sums”  $\sum$ .

He uses  $\prod_x$  to express that propositions referring to a domain  $x$  are valid for any domain  $x$  in the basic manifold 1, and  $\sum_x$  to say that the proposition is not necessarily valid for all, but at least for a certain domain  $x$ , or for several certain domains  $x$  of our manifold 1, i. e., for at least one  $x$  (Schröder 1891, § 29, 26–27).

For Schröder the use of  $\sum$  and  $\prod$  in logic is perfectly analogous to arithmetic. Existential quantifier and universal quantifier are therefore interpreted as possibly indefinite logical addition or disjunction and logical multiplication or conjunction, respectively. This is expressed by the following definition which also shows the duality of  $\sum$  and  $\prod$  (Schröder 1891, § 30, 35).

$$\sum_{\lambda=1}^{\lambda=n} a_{\lambda} = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n \quad | \quad \prod_{\lambda=1}^{\lambda=n} a_{\lambda} = a_1 a_2 a_3 \cdot \dots \cdot a_{n-1} a_n .$$

Proceeding from this interpretation Schröder can be counted among the precursors of infinitary logic (cf. Moore 1997).

## 4 Conclusions

It is obvious that Schröder had a real quantification theory, similar to that of Frege, but small wonder, Schröder's book was published in 1891, Frege's *Begriffsschrift* in 1879. So Schröder had 12 years to make himself familiar with modern quantification and to adopt it in his own theory. This is a simple and plausible answer, but it is false.

Schröder never claimed any priority for his quantification theory, but he did not take it from Frege. Schröder himself gives the credit for his use of  $\sum$  and  $\prod$  to Charles S. Peirce and Peirce's student Oscar Howard Mitchell (Schröder 1891, 27). About these contributions Geraldine Brady recently wrote (Brady 2000, 6, cf. *ibid.*, 86–94) that in 1883

one of Peirce's students, O. H. Mitchell, developed a rudimentary system for quantification, limited to a theory of quantified propositional functions with two prenex quantifiers [Mitchell 1883]. In the same year, inspired by Mitchell, Peirce introduced quantifiers as operations on propositional functions over a specific domain and part of the semantics of first-order logic for prenex formulas over this domain [Peirce 1883]. This direction of research culminated two years later in Peirce's system of first-order logic, which is expressively equivalent to our modern-day first-order logic with functions.

These contributions are later than Frege's, but they are independent from them, born in developing the Boolean kind of logic.

Therefore, quantification theory evidently cannot be the criterion for distinguishing the two big traditions in the history of logic!

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