Hilbert's Paradox*

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Abstract

In diesem Aufsatz wird erstmals die Hilbertsche Antinomie vollständig publiziert. David Hilbert hat sie während seiner Auseinandersetzungen mit der Cantorschen Mengenlehre gefunden. Seinen Angaben zufolge wurde Ernst Zermelo durch sie zu seiner Version der Zermelo-Russellschen Antinomie angeregt. Es handelt sich um die Antinomie der Menge aller durch Addition (Vereinigung) und Selbstbelegung erzeugbaren Mengen. Sie ähnelt der Cantorschen Antinomie der Menge aller Kardinalzahlen, ist aber, so Hilbert, "rein mathematisch", da in ihr ein offensichtlicher Bezug zur Cantorschen Kardinal- und Ordinalzahlarithmetik vermieden wird.

In this paper Hilbert's paradox is for the first time published completely. It was discovered by David Hilbert while struggling with Cantor's set theory. According to Hilbert, it initiated Ernst Zermelo's version of the Zermelo-Russell paradox. It is the paradox of all sets derived from addition (union) and self-mapping. It is similar to Cantor's paradox of the set of all cardinals, but, following Hilbert, of "purely mathematical nature", because an open reference to Cantor's cardinal and ordinal arithmetic is avoided.

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1 Introduction

In 1903 Gottlob Frege published the second volume of his *Grundgesetze der Arithmetik* [Frege 1903] containing the admission that the logical system

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used there for the foundation of arithmetic had proved to be inconsistent. He sent a copy of this volume to David Hilbert, who thanked him in a letter dated 7 November 1903. In this letter Hilbert referred to Frege's description of Russell's paradox in the postscript, and wrote that "this example" was already known in Göttingen. In a footnote he added "I believe Dr Zermelo discovered it three or four years ago after I had communicated my examples to him" and continued

> I found other even more convincing contradictions as long as four or five years ago; they led me to the conviction that traditional logic is inadequate and that the theory of concept formation needs to be sharpened and refined.[1]

Hence, Hilbert maintained that he had formulated logical paradoxes around 1898 or 1899 which he communicated to Zermelo, thereby initiating Zermelo's independent discovery of Russell's paradox which took place around 1899 or 1900.

Zermelo's part in this story is well-known, Hilbert's role, however, remains almost completely obscure. Hilbert never published a new paradox. There is no paradox associated to Hilbert in standard catalogues of paradoxes. What could it be? What could be more convincing than Russell's paradox?

In this paper we present a candidate for Hilbert's paradox. In the first part we give evidence for our suggestion and provide the historical context. In the second part Hilbert's paradox is described and its systematic significance is discussed.

Throughout the paper we use the term "paradox", bearing in mind, however, that as early as 1907 Ernst Zermelo had suggested to use "antinomy" instead. After having read the proof sheets of the paper "Bemerkungen zu den Paradoxieen von Russell und Burali-Forti" co-authored by his student Kurt Grelling and his philosophical colleague in Göttingen Leonard Nelson [Grelling/Nelson 1908], he criticized in a comment to Nelson the use of the term "paradox", "antinomy" being much more precise. "Paradox" means, he wrote, "a statement contradicting the *common opinion*, it doesn't contain anything of an *inner contradiction* (as is the case for the paradoxes of Russell and Burali-Forti, and expressed by the term "antinomy").[2]

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[1] [Frege 1980, p. 51]. German original [Frege 1976, pp. 79–80].

[2] Zermelo's postcard to Leonard Nelson, Glion (Switzerland), no date (postmark 22 December 1907): "Wollen Sie nicht auch lieber 'Antinomie' sagen, statt 'Paradoxie', da der erstere Ausdruck sehr viel präziser ist." A month later Zermelo wrote to Nelson in a postcard, Glion, no date (postmark 20 January 1908): "Das Wort 'Paradoxie' scheint mir von Hessenberg [Gerhard Hessenberg, co-editor of the new series of the *Abhandlungen der Fries'schen Schule*, where the joint paper was published] weil es eben etwas *ganz anderes* bedeutet, nämlich eine Aussage, welche der *herkömmlichen Meinung* widerstreitet; von

2 Historical Context

2.1 Zermelo's Paradox

We turn to Zermelo's part in this story. Zermelo came to Göttingen in 1897 in order to work for his *Habilitation*. His special fields of competence were the calculus of variations and mathematical physics, such as thermodynamics and hydrodynamics.[3] Under the influence of Hilbert he changed his focus of interests to set theory and foundations. He became Hilbert's collaborator in the foundations of mathematics, a first member of Hilbert's school before it was even established. Zermelo's first set-theoretical publication on the addition of transfinite cardinals dates from 1902 [Zermelo 1902], but as early as the winter term 1900/1901 he gave a lecture course on set theory in Göttingen. It is possible that he found the paradox while preparing this course. He referred to it in the famous polemical paper "A New Proof of the Possibility of a Well-Ordering" of 1908 [Zermelo 1908a]. There Zermelo noted that he had found the paradox independently of Russell, and that he had mentioned it to Hilbert and other people already before 1903, the year when it was first published by Frege and Russell ([Frege 1903], [Russell 1903]). And indeed, among the papers of Edmund Husserl, until 1916 professor of philosophy in Göttingen, a note in Husserl's hand was found, partially written in Gabelsberger shorthand, saying that Zermelo had informed him on 16 April 1902 that the assumption of a set M that contains all of its subsets m, m', ... as elements, is an inconsistent set, i. e., a set which, if treated as a set at all, leads to contradictions.[4] Zermelo's message was a comment on a review that Husserl had written on the first volume of Ernst Schröder's *Vorlesungen über die Algebra der Logik* [Schröder 1890]. Schröder had criticized George Boole's interpretation of the symbol 1 as the class of everything that can be a subject of discourse (the universe of discourse, universal class).[5] Husserl had dismissed Schröder's argumentation as sophistical [Husserl 1891, p. 272], and was now advised by Zermelo that Schröder was right concerning the matter,

einem *inneren Widerspruch* enthält es gar nichts," Archiv der sozialen Demokratie, Bonn, Nelson papers.

[3] On Zermelo's activities in Göttingen cf. esp. [Moore 1982], [Peckhaus 1990a, pp. 76–122], [Peckhaus 1990b].

[4] Critical edition in *Husserliana* XXII [Husserl 1979, p. 399]: "*Zermelo* teilt mit (16. April 1902) [...] Eine Menge M, welche *jede* ihrer Teilmengen m, m′ ... *als Element* enthält, ist eine inkonsistente Menge, d. h. eine solche Menge, wenn sie überhaupt als Menge behandelt wird, führt zu Widersprüchen." English translation in [Rang/Thomas 1981].

[5] [Schröder 1890, p. 245]: "Es [ist] in der That unzulässig [...], unter 1 eine so umfassende, sozusagen ganz offene Klasse, wie das oben geschilderte 'Universum des Diskussionsfähigen' (von *Boole*) zu verstehen." Schröder referred to Boole's definition of the universe of discourse and his interpretation of the symbol 1, cf. [Boole 1854, pp. 42–43].

but not in his proof.

The document from the Husserl papers provides convincing evidence for Hilbert's assertion concerning Zermelo. It is furthermore confirmed by Zermelo's own recollections. In 1936, Heinrich Scholz was working on the papers of Gottlob Frege which he had acquired for his department at the University of Münster. He had found Hilbert's letter to Frege, mentioned above, and now asked Zermelo what paradoxes Hilbert referred to in this letter.[6] Zermelo answered that the set-theoretic paradoxes were often discussed in the Hilbert circle around 1900, and he himself had given at that time a precise formulation of the paradox which was later named after Russell.[7]

2.2 Traces of Hilbert's Paradox

But what about Hilbert's own paradox? It left some traces in history. The most prominent one is Otto Blumenthal's hint in his biography of Hilbert published in the third volume of Hilbert's *Collected Works* [Blumenthal 1935]. There Blumenthal mentions the paradoxes of set theory and relates them to the second of Hilbert's problems presented in the famous Paris problems lecture in 1900 [Hilbert 1900a], i. e., the problem of proving the consistency of the axioms of arithmetic. According to Blumenthal the paradoxes showed that certain operations with the infinite, which everyone thought to be allowed, led unquestionably to contradictions. Blumenthal reports that Hilbert convinced himself of this fact by constructing the example of an inconsistent set of all sets resulting from union and self-mapping, i. e., purely mathematical operations [Blumenthal 1935, pp. 421–422].

Another trace can be found in the year 1907. The Göttingen philosopher Leonard Nelson and the student of mathematics and philosophy Kurt Grelling were working on one of the first philosophical papers to discuss the paradoxes, here especially the ones of Russell and Burali-Forti [Grelling/Nelson 1908]. The joint paper contained a general formulation suitable for several paradoxes, among them the semantical "heterological paradox" or "Grelling's paradox" (cf. [Peckhaus 1990a, pp. 168–195], [Peckhaus 1995]). From a letter of the Göttingen mathematician Ernst Hellinger to Leonard Nelson, dated 28 December 1907,[8] we learn that Hellinger had read a manuscript version of the

[6]Heinrich Scholz to Zermelo, dated Münster, 5 April 1936, University Archive Freiburg i. Br., Zermelo papers, C 129/106.

[7]Zermelo to Scholz, dated Freiburg i. Br., 10 April 1936, Institut für mathematische Logik und Grundlagenforschung, Münster, Scholz papers: "Über die mengentheoretischen Antinomien wurde um 1900 herum im Hilbert'schen Kreise viel diskutiert, und damals habe ich auch der Antinomie von der größten Mächtigkeit die später nach Russell benannte präzise Form (von der 'Menge aller Mengen, die sich nicht selbst enthalten') gegeben. Beim Erscheinen des Russellschen Werkes [. . .] war uns das schon geläufig."

[8]Hellinger to Nelson, dated Breslau, 28 December 1907, Archiv der sozialen Demokratie,

paper. He suggested to add a note on Hilbert's paradox, because its appearance was more mathematical and perhaps more suitable for mathematicians not working in set theory. In the end Hilbert's paradox was not included, because Grelling failed to reduce it to the general formulation. Nevertheless we can state that, at least in Göttingen, Hilbert's paradox was generally known in that time.

2.3 Hilbert and Cantor

Given the time period referred to by Hilbert, it can be assumed that Hilbert formulated the paradox during his discussions with Georg Cantor, documented in their correspondence between 1897 and 1900.[9] The main topics were Cantor's problems with the assumption of a set of all cardinals. Already in the first of Cantor's letters to Hilbert, dated 26 September 1897 [Cantor 1991, no. 156, pp. 388–389], Cantor proves that the totality of alephs does not exist, i. e., that this totality is not a well-defined, finished set [*fertige Menge*]. If it is taken to be a finished set, a certain larger aleph would follow on this totality. So this new aleph would at the same time belong to the totality of all alephs, and not belong to it, because of being larger than all alephs (ibid., p. 388). Cantor consequently distinguished sets from other kinds of multiplicities, i. e., "finished" sets from multiplicities which are not sets, like the totality of all cardinals. The latter multiplicities are "absolutely infinite", unlike the former ones, the "transfinite" sets. In a later letter Cantor gave the following characterization of a finished set: A set can be imagined as finished if it is consistently possible to imagine all of its elements as being gathered, the set itself therefore as *one* compound thing, i. e., if it is possible to imagine the totality of its elements as existing.[10] This is, however, impossible for the absolute infinite which he identifies with God. The *absolute infinite* doesn't allow any determination [Cantor 1883, p. 556]. Realized in its highest perfection in God it has to be strictly opposed to the *actual infinite* which he calls the transfinite [Cantor 1887, pp. 81–82].

It is well-known that Cantor later changed his terminology. In May 1899

Bonn, Nelson papers: "Es wäre vielleicht nicht unzweckmäßig, es [Hilbert's paradox] zu erwähnen, da es mathematischer aussieht als die andern, und vielleicht auch dem nichtmengentheoretischen Mathematiker sympathischer aussieht, als das W-Paradoxon [i. e., Burali-Forti's paradox of the set W of all ordinals]."

[9] For a comprehensive discussion of this correspondence cf. [Purkert/Ilgauds 1987, 147–166]. Extracts are published in [Cantor 1991]. For Cantor's reaction to the paradox see also [Ferreirós 1999, 290–296].

[10] Cantor's letter to Hilbert, dated 2 October 1897 [Cantor 1991, p. 390], also published in [Purkert/Ilgauds 1987, no. 44, pp. 226–227]. A similar definition can be found in Cantor's letter to Hilbert, Halle, 10 October 1898 [Cantor 1991, no. 158, 396–397, definition on p. 396].

he wrote to Hilbert that he had become accustomed to replace what he formerly had called "finished" by the expression "consistent". The notion "sets" stood now for "consistent multiplicities".[11]

Cantor disproves the existence of the totality of all cardinals by showing that the assumption of its existence contradicts his definition of a set as a comprehension of certain well distinguished objects of our intuition or our thinking in a whole.[12] The totality of all cardinals (and of all ordinals) cannot be thought of as *one* such thing, contrary to actual infinite objects like transfinite sets. He is therefore not really concerned with paradoxes and their solution, but with non-existence proofs using *reductio-ad-absurdum* arguments.[13]

From these passages we learn that Hilbert was concerned with what was later called "Cantor's paradox", i. e., the paradox of the greatest cardinal, or of the set of all cardinals. It is clear, however, that the contradiction discussed by Cantor served only as a paradigmatic example for other inconsistent multiplicities, i. e., totalities resulting from unrestricted comprehension. Nevertheless, there is no evidence that Cantor and Hilbert discussed the contradiction resulting from the assumption of a greatest ordinal, today known as "Burali-Forti's paradox", although this has been claimed by several authors.[14] Usually Cantor's letter to Philip E. B. Jourdain of 4 November 1903 is taken as evidence for Cantor having known the paradox of the greatest ordinal before its publication by Cesare Burali-Forti [Burali-Forti 1897], and that he had communicated this paradox to Hilbert as early as 1896.[15] In fact Cantor showed in this letter to Jourdain that the assumption of a system of all ordinals leads to a contradiction. In his communication with Hilbert of 9 May 1899, however, he only referred to the assumption of a greatest cardinal.[16] Purkert and Ilgauds made it furthermore plausible [Purkert/Ilgauds

[11] Cantor's letter to Hilbert, Halle, 9 May 1899 [Cantor 1991, no. 160, p. 399].

[12] [Cantor 1895/97], quoted in [Cantor 1932, p. 282]: "Unter einer 'Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) zu einem Ganzen."

[13] We follow in this evaluation [Moore/Garciadiego 1981], [Garciadiego Dantan 1992].

[14] E. g., [Fraenkel 1930, pp. 261], [Meschkowski 1983, p. 144].

[15] The letter was quoted by Jourdain [Jourdain 1904] and mentioned by Felix Bernstein [Bernstein 1905, 187]. Gerhard Hessenberg referred to Bernstein when maintaining Cantor's priority [Hessenberg 1906, § 98, p. 631]. From there it became standard folklore. Cf. [Grattan-Guinness 2000, pp. 117–119].

[16] Cantor's letter to Philip E. B. Jourdain, dated Halle 4 November 1903 [Cantor 1991, no. 172, pp. 433–434, quote p. 433]: "Den unzweifelhaft richtigen Satz, daß es außer den Alephs keine anderen transfiniten Cardinalzahlen giebt, habe ich vor über 20 Jahren (bei der Entdeckung der Alephs selbst) intuitiv erkannt. [...] Schon vor 7 Jahren machte ich Herrn Hilbert, vor 4 Jahren Herrn Dedekind darauf bezügliche briefliche Mitteilung." The extensive correspondence between Cantor and Jourdain is published in [Grattan-Guinness

1987, p. 151] that Cantor's recollections were erroneous. He most probably referred to his letter to Hilbert of 26 September 1897, mentioned above. The notion of the greatest ordinal was also the topic of a letter Cantor wrote to Dedekind on 3 August 1899. There he proved that the system Ω of all numbers is an inconsistent, absolutely infinite multiplicity.[17] In this letter Cantor also referred to the totality of everything imaginable ("Inbegriff alles Denkbaren"), i. e., Dedekind's own assumption in *Was sind und was sollen die Zahlen?* [Dedekind 1888], needed to prove that there are infinite systems (sets).[18] Cantor showed that his non-existence proofs also hold with this assumption.

Hilbert's responses in correspondence have not been preserved,[19] but he published his opinion at prominent places. In the paper "On the Concept of Number" from 1900 [Hilbert 1900b], Hilbert's first paper on the foundations of arithmetic, he gave a set of axioms for arithmetic, and claimed that only a suitable modification of known methods of inference would be needed for proving the consistency of the axioms. If this proof were successful, the existence of the totality of real numbers would be shown at the same time. In this context he referred to Cantor's problem of whether the system of real numbers is a consistent, or finished, set. He stressed:

> Under the conception above, the doubts which have been raised against the existence of the totality of all real numbers (and against the existence of infinite sets in general) lose all justification; for by the set of real numbers we do not have to imagine the totality of all possible laws according to which the elements of a fundamental sequence can proceed, but rather—as just described—a system of things whose internal relations are given by a *finite and closed* set of axioms [. . .], and about which new statements are valid only if one can derive them from the axioms by means of a finite number of logical inferences.[20]

He also claimed that the existence of the totality of all powers or of all Cantorian alephs could be disproved, i. e., in Cantor's terminology, that the system of all powers is an inconsistent (not finished) set (ibid.).

1972–73].

[17] Cantor to Dedekind, dated Halle, 3 August 1899, [Cantor 1991, no. 163, pp. 407–411]. It is one of the best known of Cantor's letters, published already in Zermelo's edition of Cantor's collected works [Cantor 1932, pp. 443–447]. Ivor Grattan-Guinness has shown, however, that Zermelo combined this letter with the one of 28 July 1899 and even changed the original wording at some places [Grattan-Guinness 1974–75]. The correct text of the letter of 28 July 1899 is found in [Cantor 1991, no. 162, p. 405].

[18] [Dedekind 1888, p. 14]: "Meine Gedankenwelt, d. h. die Gesamtheit S aller Dinge, welche Gegenstand meines Denkens sein können, ist unendlich."

[19] In his letter to Hilbert of 2 October 1897 Cantor referred to some of Hilbert's objections, quoted in [Purkert/Ilgauds 1987, pp. 226–227].

[20] [Hilbert 1996b, p. 1095]. German original [Hilbert 1900b, p. 184].

Hilbert took up this topic again in his famous Paris lecture on "Mathematical Problems".[21] In the context of his commentary on the second problem concerning the consistency of the arithmetical axioms he used the same examples from Cantorian set theory and the continuum problem as in the earlier lecture. "If contradictory attributes be assigned to a concept," he wrote, "I say, that mathematically the concept does not exist" [Hilbert 1996a, p. 1105].

According to Hilbert a suitable axiomatization would be able to avoid the contradictions resulting from the attempt to comprehend absolute infinite multiplicities as units, because only those concepts had to be accepted which could be derived from an axiomatic base.

2.4 The 1905 lecture

Although it is evident that Hilbert was at that time deeply concerned with the problems of set theory, we have found no direct evidence that Hilbert had formulated contradictions in this context, or even a paradox of his own. Indirect evidence can be found, however, in documents dating from a few years later.

Only after the publication of the paradoxes by Russell and Frege, and especially through Frege's reaction, the logical significance of this kind of contradiction became evident.[22] Now mathematicians understood that these paradoxes were not the simple contradictions that they were familiar with in their everyday *reductio ad absurdum* arguments. As logical paradoxes they seriously affected Hilbert's axiomatic programme, especially the proposed consistency proof for arithmetic. It is a matter of course that a consistency proof, based on a logic proved to be inconsistent, could not be given. Hilbert first expressed this new insight in a talk delivered at the Third International Congress of Mathematicians in Heidelberg in August 1904 [Hilbert 1905c]. In this lecture "On the Foundations of Logic and Arithmetic" he demanded a "partly simultaneous development of the laws of logic and arithmetic" [Hilbert 1905c, p. 176]. According to Blumenthal [Blumenthal 1935, p. 422], this lecture remained completely misunderstood and several of Hilbert's ideas proved to be defective. Nevertheless it was the first step in the construction of a foundational system of mathematics avoiding the paradoxes.

The next step was taken in a lecture course on the "Logical Principles of Mathematical Thinking" which Hilbert gave in Göttingen in the summer term of 1905. Two sets of notes of this lecture course were preserved. The "official" notes are from Ernst Hellinger, then a student of mathematics. They contain marginal notes in Hilbert's hand [Hilbert 1905a]. Another set

[21] [Hilbert 1900a], English translations [Hilbert 1902], [Hilbert 1996a].

[22] Cf. [Moore 1978], [Moore 1980, pp. 104–105], [Moore/Garciadiego 1981], [Garciadiego Dantan 1992].

was produced by the student of mathematics and physics Max Born [Hilbert 1905b]. Part B of these notes, on "The Logical Foundations", starts with a comprehensive discussion of the paradoxes of set theory. It begins with metaphorical considerations on the general development of science:

> It was, indeed, usual practice in the historical development of science that we began cultivating a discipline without many scruples, pressing onwards as far as possible, that we thereby, however, then ran into difficulties (often only after a long time) that forced us to turn back and reflect on the foundations of the discipline. The house of knowledge is not erected like a dwelling where the foundation is first well laid-out before the erection of the living quarters begins. Science prefers to obtain comfortable rooms as quickly as possible in which it can rule, and only subsequently, when it becomes clear that, here and there, the loosely joined foundations are unable to support the completion of the rooms, science proceeds in propping up and securing them. This is no shortcoming but rather a correct and healthy development.[23]

Although contradictions are quite common in science, Hilbert continued, in the case of set theory they seem to be different, because there they have a tendency towards the side of theoretical philosophy. In set theory the common Aristotelian logic and its standard methods of concept formation were used without hesitation. And these standard tools of purely logical operations, especially the subsumption of concepts under a general concept, proved to be responsible for the new contradictions.

Hilbert elucidated these considerations by presenting three examples. The first paradox discussed is the Liar paradox. The third one is "Zermelo's paradox," as the Russell-Zermelo paradox was called in Göttingen at that time. Hilbert described this paradox as purely logical, assuming that it might be more convincing for non-mathematicians. He stressed, however, that it was derived from his own paradox, the second one in his list of examples, and this second paradox was, according to Hilbert, of purely mathematical nature.[24] Hilbert expressed his opinion that this paradox

> appears to be especially important; when I found it, I thought in the beginning that it causes invincible problems for set theory that would finally lead to the latter's eventual failure; now I firmly believe, however, that everything essential can be kept after a revision of the foundations, as always in science up to now. I have not published this

[23] [Hilbert 1905b, p. 122], published in [Peckhaus 1990, p. 51].

[24] [Hilbert 1905a, p. 210]: "Als drittes Beispiel dieser Widersprüche stelle ich neben diesen meinen rein mathematischen noch einen rein logischen, den Dr. *Zermelo* aus jenem herausgezogen hat [. . .]."

> contradiction, but it is known to set theorists, especially to G. Cantor.[25]

This paradox, arising from uniting sets and mapping them to themselves, is exactly the one Blumenthal referred to in his biography. It is most likely the one Hilbert himself referred to in his letter to Frege.

3 Hilbert's Paradox

3.1 Hilbert's Presentation

The full text of Hilbert's paradox is given in the appendices, both in English translation (appendix I) and in the German original (appendix II). Here, we reconstruct the main steps of Hilbert's argument.

The paradox is based on a special notion of set which Hilbert introduces by means of two set formation principles starting from the natural numbers. The first principle is the *addition principle*. In analogy to the finite case, Hilbert argued that the principle can be used for uniting two sets together "into a new conceptual unit [...], a new set that contains each element of either sets." This operation can be extended: "In the same way, we are able to unite several sets and even infinitely many into a union." The second principle is called the *mapping principle*. Given a set $\mathcal{M}$, he introduces the set $\mathcal{M}^{\mathcal{M}}$ of *self-mappings* of $\mathcal{M}$ to itself.[26] A self-mapping is just a total function which maps the elements of $\mathcal{M}$ to elements of $\mathcal{M}$.[27]

Now, he considers all sets which result from the natural numbers "by applying the operations of addition and self-mapping an arbitrary number of times." By use of the addition principle which allows to build the union of arbitrary sets one can "unite them all into a sum set $\mathcal{U}$ which is well-defined." In the next step the mapping principle is applied to $\mathcal{U}$, and we get $\mathcal{F} = \mathcal{U}^{\mathcal{U}}$ as the set of all self-mappings of $\mathcal{U}$. Since $\mathcal{F}$ was built from the natural numbers by using the two principles only, Hilbert concludes that it has to be contained in $\mathcal{U}$. From this fact he derives a contradiction.

Since "there are 'not more' elements" in $\mathcal{F}$ than in $\mathcal{U}$ there is an assignment of the elements u_i of $\mathcal{U}$ to elements f_i of $\mathcal{F}$ such that all elements of f_i are used. Now one can define a self-mapping g of $\mathcal{U}$ which differs from all

[25] [Hilbert 1905a, p. 204], published in [Peckhaus 1990, p. 52].

[26] Hilbert used the German term "*Selbstbelegung*" which is translated here by "self-mapping". The term "*Belegung*" was already used by Cantor [Cantor 1895/97, § 4, p. 486 (1895)], cf. also [Cantor 1932, p. 287]. In his edition of Georg Cantor's collected works Zermelo explained *Belegung* as a function with explicitly given domain and (potential) range [Cantor 1932, footnote [3], p. 352].

[27] In classical logic, $\mathcal{M}^{\mathcal{M}}$ is isomorphic to $2^{\mathcal{M}}$, and the set of all mappings from $\mathcal{M}$ to $\{0,1\}$ is isomorphic to $\mathcal{P}(\mathcal{M})$, the power set of $\mathcal{M}$.

f_i. Thus, g is not contained in $\mathcal{F}$. Since $\mathcal{F}$ was assumed to contain all self-mappings we have a contradiction. In order to define g Hilbert used Cantor's diagonalization method. If f_i is a mapping u_i to $f_i(u_i) = u_{f_i^{(i)}}$ he chooses an element $u_{g^{(i)}}$ different from $u_{f_i^{(i)}}$ as the image of u_i under g. Thus, we have $g(u_i) = u_{g^{(i)}} \neq u_{f_i^{(i)}}$ and g "is distinct from any mapping f_k of $\mathcal{F}$ in at least one assignment."[28]

Hilbert finishes his argument with the following observation:

> We could also formulate this contradiction so that, according to the last consideration, the set $\mathcal{U}^{\mathcal{U}}$ is always bigger [of greater cardinality][29] than $\mathcal{U}$ but, according to the former, is an element of $\mathcal{U}$.

3.2 Brief Reconstruction

In order to make the argument more comprehensible, the paradox can be presented in the following way. First we define a notion of set:

Definition 1 *We define inductively:*

1. *The natural numbers as a whole are a set.*[30]
2. Addition principle*: If we have an arbitrary, possibly infinite collection of sets, the union of all these sets is a set.*
3. Mapping principle*: The totality of all total functions from a given set into itself is a set.*

Now we take the closure of all sets introduced according to the following definition (this union is well defined according to the addition principle):

Definition 2 *Let $\mathcal{U}$ be the union of all sets defined according to definition 1.*

Now we can apply the mapping principle to it.

Definition 3 *Let $\mathcal{F}$ be the set $\mathcal{U}^{\mathcal{U}}$.*

Obviously $\mathcal{F}$ is built according to our definition of sets. We have used the addition principle to define $\mathcal{U}$ and then the mapping principle to define $\mathcal{F}$. But that means, $\mathcal{F}$ has to be contained in $\mathcal{U}$ because $\mathcal{U}$ was the union of all sets built according to the definition of sets. Thus, we get the following

[28] Hilbert's notation $u_{g^{(i)}}$ is somewhat clumsy. In fact, it is enough to say that $g(u_i) = v_i$ for an element v_i of $\mathcal{U}$ with $v_i \neq f_i(u_i)$.

[29] Remark later added in Hilbert's hand in Hellinger's lecture notes.

[30] Hilbert even argues that the natural numbers can be defined from finite sets using the addition principle.

Lemma 4 $\mathcal{F} \subseteq \mathcal{U}$.

From this lemma it follows that there exists a function of $\mathcal{U}$ in $\mathcal{F}$ whose range is the whole set $\mathcal{F}$. Therefore, we can apply Cantor's diagonalization method to define a function from $\mathcal{U}$ to $\mathcal{U}$ which is distinct from each element of $\mathcal{F}$.

Proposition 5 *There exists a total function g from $\mathcal{U}$ to $\mathcal{U}$ such that $g \notin \mathcal{F}$.*

But by definition of $\mathcal{F}$, this set contains *all* total function from $\mathcal{U}$ to $\mathcal{U}$. Thus, we get as a

Corollary 6 *The system of sets defined by 1 is contradictory.*

3.3 Analysis of the Paradox

The reconstruction given above reveals the source of the paradox. Obviously the addition principle is too vague. Hilbert allows "to unite several sets and even infinitely many into a union," he even allows to "unite them all," i. e., all sets defined by addition and self-mapping. He does not determine, however, the domain of the universal quantifier. The definition of the set $\mathcal{U}$ is, thus, based on an *impredicative construction*, because $\mathcal{U}$ itself has to belong to this domain. In short: The definition of $\mathcal{U}$ depends on a totality containing $\mathcal{U}$ itself.

These problems can be overcome by restricting the addition principle. It has to be demanded that the sets united have to be elements of another set already established. And this is, in fact, the way in which Zermelo proceeded in his axiomatization of set theory. This axiomatic system, refined by Fraenkel and Skolem and called ZFC, is still today generally accepted as *the* basis of mathematics. In ZFC we have a *union axiom* corresponding to the addition principle. But in contrast to the addition principle, a *family of sets T being itself a set* is demanded which can be regarded as an *index set* giving some control over the sets gathered in the union [Zermelo 1908b, 265]. Nowadays, the union axiom is stated as:

$$\forall T \exists S \forall x (x \in S \leftrightarrow \exists U (x \in U \land U \in T))$$

Fraenkel correctly saw that an unrestricted union axiom within axiomatized set theory led to the same problems as the ones connected with Russell's paradox. He saw the reason for these problems in an unconcerned use of the notion "arbitrarily many." Fraenkel referred directly to the union axiom, so his analysis reads like a diagnosis of the cause of Hilbert's paradox.[31]

[31] [Fraenkel 1927, p. 71]: "Will man [...] zu etwas allgemeineren Prozessen [of set formation] fortschreiten, so muß man [...] auch die Zusammenfassung der *Elemente ver-*

Although Hilbert worked only in a restricted domain of sets, containing only those sets formed by addition and self-mapping, his addition principle was itself too vague, so that it resulted in effects similar to those of Cantor's comprehension.[32] From another perspective the lack of a proper quantification theory is conspicuous. Hilbert's formulation is therefore affected by the general problems of impredicativity.

Zermelo's axiomatization of set theory can thus be read as an answer to two different paradoxes. His strategy was to avoid unrestricted comprehension, leading to Cantor's paradox (and also to the Zermelo-Russell paradox), and unrestricted union, leading to Hilbert's paradox. He easily prevented the formulation of Hilbert's paradox by introducing the family set T in the union axiom (axiom V). The paradoxes resulting from unrestricted comprehension were avoided by introducing the separation axiom (axiom III) which ensures that each set M has at least one subset M_0 not being element of M [Zermelo 1908b, 264].

In contrast to the addition principle, the mapping principle is "innocent" of the emergence of Hilbert's paradox. If we replace the total functions from $\mathcal{M}$ to $\mathcal{M}$ by total functions from $\mathcal{M}$ to the set $\{0, 1\}$ we get the set of charateristic functions of all subsets of $\mathcal{M}$. Thus, the mapping principle is closely related to the power set axiom as it is used in modern set theory. Hilbert demanded for the mapping principle that the set of all self-mappings is obtained over *sets* already established, a restriction also valid for the modern power set axiom.

schiedener Mengen anstreben. Einen Fingerzeig, wie dies zu erfolgen hat, liefert uns die Bildung der Vereinigungsmenge in der CANTORschen Mengenlehre, wo die sämtlichen Elemente beliebig vieler Mengen zu einer neuen Menge, der Vereinigungsmenge, vereinigt werden können [...]. Hinsichtlich der gefahrdrohenden Folgen eines unbekümmerten Gebrauchs des Begriffs 'beliebig viele' sind wir freilich, z. B. durch das RUSSELLsche Paradoxon, hinlänglich gewitzigt; wir gehen daher nicht wie früher von beliebig vielen Mengen aus, sondern setzen voraus, daß diese Mengen als die Elemente einer bereits als legitim erkannten Menge säuberlich gegeben sind."

[32]This is also the conclusion of Paul Bernays who reported in 1971, obviously referring to Hilbert's paradox: "Der Gedanke der Beschränkung auf solche Mengen, die man, beginnend mit einer Ausgangsmenge (etwa der Menge der natürlichen Zahlen) durch Potenzmengenbildungen, Vereinigungsprozesse und Aussonderungen bilden kann, wurde—wie ich aus Erzählungen von Hilbert weiß—seinerzeit auch erwogen; er führte aber zunächst gerade zu einer Verschärfung der Paradoxien, da man die Vereinigungsprozesse nicht genügend deutlich normierte, vielmehr die Zusammenfassung der durch die angegebenen Prozesse gewinnbaren Mengen zu einer Menge ihrerseits als einen zulässigen Vereinigungsprozeß ansah" [Bernays 1971/1976, p. 199]. We would like to thank José F. Ruiz, Madrid, for bringing this quote to our attention.

4 Conclusion

Hilbert's paradox is closely related to Cantor's own paradox. Both Cantor and Hilbert construct "sets" which lead to contradictions being proved with the help of Cantor's diagonalization argument. However, the ways in which these "sets" are constructed differ essentially. According to Cantor ([Cantor 1883, § 11], cf. [Cantor 1932, pp. 195–197]), there are three principles for the generation of cardinals. The first principle ("erstes Erzeugungsprinzip") concerns the generation of real whole numbers [*reale ganze Zahlen*, i. e., ordinal numbers] by adding a unit to a given, already generated number. The second principle allows the formation of a new number, if a certain succession of whole numbers with no greatest number is given. This new number is imagined as the limit of this succession. Cantor adds a third principle, the inhibition or restriction principle ("Hemmungs- oder Beschränkungsprinzip") which grants that the second number class has not only a higher cardinality than the first number class, but exactly the next higher cardinality. Considering Cantor's general definition of a set as the comprehension of certain well-distinguished objects of our intuition or our thinking as a whole ([Cantor 1895/97], [Cantor 1932, p. 282]), one can justly ask whether the sets of all cardinals, of all ordinals or the universal set of all sets are sets according to this definition, i. e., whether an unrestricted comprehension is possible. Cantor denies this, justifying his opinion with the help of a *reductio ad absurdum* argument, but he doesn't exclude the possibility of forming the paradoxes by provisions in his formalism.

Hilbert, on the other hand, introduces two alternative set formation principles, the addition principle and the mapping principle, but they lead to paradoxes as well. In avoiding concepts from transfinite arithmetic Hilbert believes that the purely mathematical nature of his paradox is guaranteed. For him, this paradox appears to be much more serious for mathematics than Cantor's, because it concerns an operation that is part of everyday practice of working mathematicians.

The significance of Hilbert's paradox for the history of mathematics should now be obvious. The paradox shows the importance of the end 19th century discussion on universal sets and classes, e. g., Cantor's absolutely infinite totalities, Dedekind's infinite totality of all things which might become objects of our thinking, and Boole's universe of discourse. From the beginning the limitation of size argument played a role (cf. [Hallett 1984]). This discussion marked a latent foundational crisis in mathematics. The mathematicians involved were dealing with paradoxes, i. e., contradictions that are, they believed, avoidable. The foundational crisis became manifest in 1903, when Bertrand Russell and Gottlob Frege published the insight that "Russell's paradox" could be derived from Frege's system of the *Grundgesetze*. Now mathematicians were dealing with antinomies, i. e., intrinsic contradictions that could not easily be solved. Even this new move was closely connected to the earlier discussion because Russell found his own paradox while investigating Cantor's set theory (cf. [Garciadiego Dantan 1992], [Grattan-Guinness 1978], [Grattan-Guinness 2000, pp. 310–315], [Moore 1980, pp. 104–105]). Hilbert himself had to change his axiomatic programme. Now logic and set theory moved into the focus of his foundational research (cf. [Peckhaus 1990, pp. 61–75]).

Appendix I: Hilbert's Paradox (English Translation)

[Marginal note: 18th lecture, 10 July] [...] |[204] In addition, I now come to two examples of contradictions which are much more convincing, the first, being of purely mathematical nature, appears to be especially important; when I found it, I thought in the beginning that it causes invincible problems for set theory that would lead to the latter's eventual failure; now I firmly believe, however, that everything essential can be kept after a revision of the foundations, as always in science up to now. I have not published this contradiction, but it is known to set theorists, especially to G. Cantor. Anyhow, we regard finite sets, represented by finitely many numbers, as the operational basis permitted, and also the countable infinite set $1, 2, 3 \ldots$ of all natural numbers. Furthermore, it seems to be allowed to unite two such sets $(1, 2, 3 \ldots)$ and $(a_1, a_2, a_3 \ldots)$ into a new conceptual unit $(1, 2, 3 \ldots, a_1, a_2, a_3 \ldots)$, i. e., a new set that contains each element of either sets. In the same way, we are able to unite several sets and even infinitely many into a union. We designate this as the *addition principle*, and write |[205] in short for the set obtained from $\mathcal{M}_1, \mathcal{M}_2 \ldots$,

$$\mathcal{M}_1 + \mathcal{M}_2 + \cdots$$

These unions are operations, generally applied in logic in even much more complicated cases without any hesitation. Therefore, it seems to be possible to apply them here without further ado. Besides this addition principle, we use a further consideration for forming new sets. Let $y = f(x)$ be a number theoretic function which maps to every integer value x an integer y; in a sense immediately to be understood, we can designate such a function as a *mapping* [*Belegung*] of the number sequence to itself, by imagining for instance a scheme:

$$\begin{aligned} x &= 1, 2, 3, 4 \ldots \\ y &= 2, 3, 6, 9 \ldots \end{aligned}$$

The system of all these number theoretic functions $f(x)$, or of all possible mappings of the number sequence to its own elements, forms a new set "resulting from the number sequence $\mathcal{M}$ by *self-mapping*," we write it $\mathcal{M}^{\mathcal{M}}$.|[206] As a principle following from the laws of uniting in ordinary logic and, according to it, completely unobjectionable, we can now regard the opinion that in every case well-defined sets arise from well-defined sets by the self-mapping operation (*mapping principle*). For instance, by using this principle, from the continuum of all real numbers results the set of all real functions. We want to use only these two principles unobjectionable according to all previous mathematics and logic.

We start with all finite sets of numbers and the infinite series $1, 2, 3 \ldots$ of natural numbers already derived therefrom by addition, and take all sets resulting from them by applying the operations of addition and self-mapping an arbitrary number of times; these sets form again a well-defined unit, for according to the addition principle I unite them all into a sum set $\mathcal{U}$ which is well-defined. If I form now the set $\mathcal{F} = \mathcal{U}^{\mathcal{U}}$ of self-mappings of $\mathcal{U}$, this set arises from the original number sequence via the two operations of addition and |[207] self-mapping only; it, therefore, also is one of the sets from whose addition $\mathcal{U}$ just resulted and, therefore, must be a subset of $\mathcal{U}$:

$$(1) \qquad \mathcal{F} \text{ is contained in } \mathcal{U}.$$

[Marginal note: 19th lecture, 11 July] I will now show that this leads to a contradiction. Let $u_1, u_2, u_3 \ldots$ be the elements of $\mathcal{U}$; then, each element f of $\mathcal{F} = \mathcal{U}^{\mathcal{U}}$ represents a mapping of $\mathcal{U}$ to itself, i. e., in a way a function, that assigns to each element u_i of $\mathcal{U}$ another $u_{f^{(i)}}$, where it is not at all necessary that the $u_{f^{(i)}}$ have to be distinct from one another; we, therefore, represent this element f most conveniently in schematic form:

$$f(u_1) = u_{f^{(1)}}, f(u_2) = u_{f^{(2)}}, f(u_3) = u_{f^{(3)}} \ldots$$

Our result (1), that $\mathcal{F}$ is contained in $\mathcal{U}$, can now be expressed more exactly in the following way: we can definitely assign to each single element u_i of $\mathcal{U}$ a f_i of $\mathcal{F}$ so that all f_i will thereby be used, maybe even repeatedly, but that, in any case, to each u_i only corresponds *exactly one* f_i; this means, obviously, nothing else than that there are "not more" elements f_i than u_i. We now consider such an assignment:|[208]

$$u_1|f_1, u_2|f_2, u_3|f_3 \ldots,$$

and from this I will form a new mapping g of $\mathcal{U}$ to itself that differs from all f_i, i. e., it is not an element of $\mathcal{F}$ because, in our assignment, all elements of $\mathcal{F}$ had to be used up; but since $\mathcal{F}$ includes all possible mappings, we have, thus, derived the contradiction. We again apply the principle of Cantor's diagonalization method. In the mapping f_1, let the element u_1 correspond to the $u_{f_1^{(1)}}$:

$$f_1(u_1) = u_{f_1^{(1)}};$$

if $u_{g^{(1)}}$ is an element different from $u_{f_1^{(1)}}$, then we construct the new mapping g which assigns u_1 to it:

$$g(u_1) = u_{g^{(1)}} \neq u_{f_1^{(1)}}.$$

We proceed further according to this principle; by the way, the designation of elements of $\mathcal{U}$ and $\mathcal{F}$ by number indices is not essential, and it should by no means insinuate that these sets are countable which is not at all the case. If u_2 is some element of $\mathcal{U}$, a mapping [*Belegung*] f_2 |[209] belongs to it in the mapping [*Abbildung*] of f to u; we look for the element $f_2(u_2) = u_{f_2^{(2)}}$, which it [the mapping f_2] assigns to u_2, choose $u_{g^{(2)}} \neq u_{f_2^{(2)}}$ and define a mapping g which assigns it to u_2:

$$g(u_2) = u_{g^{(2)}} \neq u_{f_2^{(2)}}$$

The mapping g which we obtain in this way has the scheme

$$g(u_1) = u_{g^{(1)}} \neq u_{f_1^{(1)}}, g(u_2) = u_{g^{(2)}} \neq u_{f_2^{(2)}}, g(u_3) = u_{g^{(3)}} \neq u_{f_3^{(3)}} \ldots$$

It is distinct from any mapping f_k of $\mathcal{F}$ in at least one assignment; namely, if u_k is the element (or one of these) corresponding to f_k in the mapping [*Abbildung*] of $\mathcal{F}$ to $\mathcal{U}$, then it follows from the definition of g that:

$$f_k(u_k) = u_{f_k^{(k)}} \qquad g(u_k) = u_{g^{(k)}} \neq u_{f_k^{(k)}}.$$

By this, we indeed have the contradiction that the well-defined mapping g cannot be a member of the set of all mappings. We could also formulate this contradiction so that, according to the last consideration, the set $\mathcal{U}^{\mathcal{U}}$ is always bigger [note in Hilbert's hand: "of greater cardinality"] than $\mathcal{U}$ but, according to the former, is an element of $\mathcal{U}$. This contradiction is not at all yet solved; anyway, one can see that it must depend upon the fact that the operations of uniting arbitrary sets or objects into |[210] new sets or totalities, respectively, is, nevertheless, not allowed, although it is always used in traditional logic, and although we have carefully applied it only to natural numbers and sets arising from them, i. e., to purely mathematical objects.

Appendix II: Hilbert's Paradox (German Original)

[Marginalie: 18. Vorles. 10. VII.] [...] |[204] Ich komme nun noch zu 2 Beispielen für Widersprüche, die viel überzeugender sind, der erste, der rein mathematischer Natur ist, scheint mir besonders bedeutsam; als ich ihn fand, glaubte ich zuerst, daß er der Mengentheorie unüberwindliche Schwierigkeiten in den Weg legte, an denen sie scheitern müßte; ich glaube jedoch jetzt sicher, daß wie stets bisher in der Wissenschaft, nach der Revision der Grundlagen alles Wesentliche erhalten bleiben wird. Ich habe diesen Widerspruch nicht publiciert; er ist aber den Mengentheoretikern, insbesondere G. Cantor, bekannt. Wir sehen die endlichen Mengen, durch endlich viele Zahlen repräsentiert, jedenfalls als erlaubte Operationsbasis an, und ebenso die abzählbar unendliche Menge $1, 2, 3 \ldots$ aller natürlichen Zahlen. Ferner erscheint es erlaubt, 2 solche Mengen $(1, 2, 3 \ldots)$ und $(a_1, a_2, a_3 \ldots)$ zu einer neuen Begriffseinheit $(1, 2, 3 \ldots, a_1, a_2, a_3 \ldots)$, einer neuen Menge, zusammenzufassen, die jedes Element der beiden Mengen enthält. Ebenso können wir auch mehrere Mengen und sogar unendlich viele zu einer Vereinigungsmenge zusammenfassen. Wir bezeichnen das als *Additionsprincip*, und schreiben |[205] die so aus $\mathcal{M}_1, \mathcal{M}_2 \ldots$ hervorgehende Menge kurz

$$\mathcal{M}_1 + \mathcal{M}_2 + \cdots$$

Diese Zusammenfassungen sind Processe, die man in der Logik stets ohne jedes Bedenken in noch weit komplicierteren Fällen anwendet; es scheint also, daß man auch hier ohne weiteres davon Gebrauch machen könnte. Außer diesem Additionsprincip verwenden wir noch eine weitere Betrachtung zur Bildung neuer Mengen. Es sei $y = f(x)$ eine zahlentheoretische Funktion, die zu jedem ganzzahligen Wert x ein ganzzahliges y zuordnet; in sofort zu verstehendem Sinne können wir eine solche Funktion auch als eine *Belegung* der Zahlenreihe mit sich selbst bezeichnen, indem wir etwa an ein Schema denken:

$$\begin{aligned} x &= 1, 2, 3, 4 \ldots \\ y &= 2, 3, 6, 9 \ldots \end{aligned}$$

Das System aller solcher zahlentheoretischen Funktionen $f(x)$ oder aller möglichen Belegungen der Zahlenreihe mit Elementen ihrer selbst bildet eine neue Menge, die "durch *Selbstbelegung* aus der Zahlenreihe $\mathcal{M}$ entstehende," wir schreiben sie $\mathcal{M}^{\mathcal{M}}$. Als aus den |[206] Zusammenfassungsgesetzen der üblichen Logik folgendes und nach ihr gänzlich unbedenkliches Princip können wir nun das ansehen, daß aus wohldefinierten Mengen durch Selbstbelegung immer wieder wohldefinierte Mengen entstehen. (*Belegungsprincip*). Durch dies Princip entsteht aus dem Continuum aller reellen Zahlen beispielsweise die Menge aller reellen Funktionen. Allein mit diesen beiden nach aller bisherigen Mathematik und Logik unbedenklichen Principen wollen wir arbeiten.

Wir gehen von allen endlichen Mengen von Zahlen und der aus ihnen bereits durch Addition entstehenden unendlichen Reihe $1, 2, 3 \ldots$ der natürlichen Zahlen aus, und fassen alle Mengen auf, die aus ihnen durch die beiden beliebig oft anzuwendenden Processe der Addition und Selbstbelegung entstehen; diese Mengen bilden wieder eine wohldefinierte Gesammtheit, nach dem

Additionsprincip vereinige ich sie alle zu einer Summenmenge $\mathcal{U}$, die wohldefiniert ist. Bilde ich nun die Menge $\mathcal{F} = \mathcal{U}^{\mathcal{U}}$ der Selbstbelegungen von $\mathcal{U}$, so entsteht diese auch aus der ursprünglichen Zahlenreihe lediglich durch die beiden Processe der Addition und |[207] Selbstbelegung; sie gehört also auch zu den Mengen, aus deren Addition erst $\mathcal{U}$ entstand, und muß daher ein Teil von $\mathcal{U}$ sein:

$$(1) \qquad \mathcal{F} \text{ ist in } \mathcal{U} \text{ enthalten.}$$

[Marginalie: 19. Vorles. 11. VII.] Ich zeige nun, dass dies zu einem Widerspruch führt. Es seien $u_1, u_2, u_3 \dots$ die Elemente von $\mathcal{U}$; jedes Element f von $\mathcal{F} = \mathcal{U}^{\mathcal{U}}$ repräsentiert dann eine Belegung von $\mathcal{U}$ mit sich selbst, d. h. eine Funktion gewissermaßen, die jedem Elemente u_i von $\mathcal{U}$ ein anderes $u_{f^{(i)}}$ zuordnet, wobei die $u_{f^{(i)}}$ keineswegs untereinander verschieden zu sein brauchen; wir stellen dies Element f am besten also durch ein Schema dar:

$$f(u_1) = u_{f^{(1)}}, f(u_2) = u_{f^{(2)}}, f(u_3) = u_{f^{(3)}} \dots$$

Unser Resultat (1), daß $\mathcal{F}$ in $\mathcal{U}$ enthalten ist, kann man nun näher so aussprechen: Man kann jedem Elemente u_i von $\mathcal{U}$ eines f_i von $\mathcal{F}$ eindeutig zuordnen, so daß alle f_i dabei verwendet werden, eventuell sogar mehrfach, aber immer jedem u_i nur *genau ein* f_i entspricht; das heißt ja offenbar nichts anderes, als daß es "nicht mehr" Elemente f_i gibt, als u_i. Eine solche Zuordnung betrachten wir nun: |[208]

$$u_1|f_1, u_2|f_2, u_3|f_3 \dots,$$

und daraus werde ich eine neue Belegung g von $\mathcal{U}$ mit sich selbst bilden, die von allen f_i verschieden ist, also gar nicht in $\mathcal{F}$ enthalten wäre, da ja bei unserer Zuordnung alle Elemente von $\mathcal{F}$ zur Verwendung kommen sollten; da aber $\mathcal{F}$ alle möglichen Belegungen enthält, so haben wir hier den Widerspruch. Wir wenden wieder das Princip des Cantorschen Diagonalverfahrens an. In der Belegung f_1 entspreche dem Element u_1 dasjenige $u_{f_1^{(1)}}$:

$$f_1(u_1) = u_{f_1^{(1)}};$$

ist $u_{g^{(1)}}$ ein von $u_{f_1^{(1)}}$ verschiedenes Element, so ordnen wir in der neu zu konstruierenden Belegung g dies dem u_1 zu:

$$g(u_1) = u_{g^{(1)}} \neq u_{f_1^{(1)}}.$$

Nach diesem Princip verfahren wir weiter; die Bezeichnung der Elemente von $\mathcal{U}$ und $\mathcal{F}$ durch Zahlenindices ist übrigens unwesentlich und soll nicht etwa andeuten, daß diese Mengen abzählbar sind, was keineswegs der Fall ist. Ist u_2 irgend ein Element von $\mathcal{U}$, so gehört ihm in der Abbildung von f auf u

eine Belegung f_2 |[209] zu; wir suchen das Element $f_2(u_2) = u_{f_2^{(2)}}$, das sie dem u_2 zuordnet, wählen $u_{g^{(2)}} \neq u_{f_2^{(2)}}$ und definieren eine Belegung g, die dies dem u_2 zuordnet:

$$g(u_2) = u_{g^{(2)}} \neq u_{f_2^{(2)}}$$

Die Belegung g, die wir so erhalten, hat das Schema

$$g(u_1) = u_{g^{(1)}} \neq u_{f_1^{(1)}}, g(u_2) = u_{g^{(2)}} \neq u_{f_2^{(2)}}, g(u_3) = u_{g^{(3)}} \neq u_{f_3^{(3)}} \ldots$$

Sie unterscheidet sich von jeder Belegung f_k aus $\mathcal{F}$ in mindestens einer Zuordnung; ist nämlich u_k das in der Abbildung von $\mathcal{F}$ auf $\mathcal{U}$ dem f_k entsprechende Element (oder eines derselben), so ist nach der Definition von g:

$$f_k(u_k) = u_{f_k^{(k)}} \qquad g(u_k) = u_{g^{(k)}} \neq u_{f_k^{(k)}}.$$

Wir haben damit in der Tat den Widerspruch, daß die wohldefinierte Belegung g nicht in der Menge aller Belegungen enthalten sein könnte. Wir könnten ihn auch dahin formulieren, daß gemäß der letzten Betrachtung die Menge $\mathcal{U}^{\mathcal{U}}$ stets größer [von Hilberts Hand: von grösserer Mächtigkeit] als $\mathcal{U}$ ist, nach der ersten aber in $\mathcal{U}$ enthalten. Dieser Widerspruch ist noch keineswegs geklärt; es ist wohl zu sehen, daß er jedenfalls darauf beruhen muß, daß die Operationen des Zusammenfassens irgend welcher Mengen, Dinge zu |[210] neuen Mengen, Allheiten doch unerlaubt ist, obwohl es die traditionelle Logik doch stets gebraucht, und wir es in vorsichtiger Weise stets nur auf ganze Zahlen und daraus entstehende Mengen, also auf rein mathematisches anwandten.

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