Calculus Ratiocinator vs. Characteristica Universalis? The Two Traditions in Logic, Revisited

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1 Introduction

It is a commonplace that in the development of modern logic towards its actual shape at least two directions or traditions have to be distinguished. These traditions may be called, following the model of Ivor Grattan-Guinness (*1988*), the tradition of the algebra of logic and the tradition of mathematical logic. They are represented by the developments going back to the British algebraist George Boole with his *The Mathematical Analysis of Logic* (*1847*), and, independently, to the German mathematician Gottlob Frege with his *Begriffsschrift* (*1879*)

Closely connected to this distinction is a comparative evaluation of the respective logical systems, culminating in the question who parented modern logic.[1] Some interpreters, among them Boole's biographer Desmond MacHale (*1985*, 71–72) or P. L. Heath (in *Prior, ed., 1967*, 542) have seen Boole as the father of modern logic. Others, like Robert Feys (*1957*) call his work the origin of modern logic. A few, like Bertrand Russell (*1951*, 74), even regarded him as the discoverer of pure mathematics (i. e., according to Russell's logicism, mathematical logic). (Those who like Wolfgang Lenzen plead for exchanging Boole with Leibniz (*1984*, 203) will not concern us here.) Most influential, however, have been those who did not deny that the continuous debate about questions relevant for modern logic started with Boole in the middle of the 19th century, but who questioned the scientific value of the algebraic tradition for the actual shape of logic. Arthur Prior may be named

[1]For a critical discussion of these positions see *Peckhaus 2000*.

in this context (*1949*, 171), or Michael Dummett (*1959*, 205), but above all W. V. O. Quine who, in a critical review of MacHale's biography (*Quine 1985*, *1995*) admitted that

> the avenue from Boole through Peirce to the present is one of continuous development, and this, if anything, is the justification for dating modern logic from Boole; for there had been no comparable influence on Boole from his more primitive antecedents. But logic became a substantial branch of mathematics only with the emergence of general quantification theory at the hands of Frege and Peirce. I date modern logic from there.

And, he continued: 'Frege got there first.' Reason enough for having 'hailed Frege as the founder of modern logic, and viewed Boole, De Morgan, and Jevons as forerunners. John Venn and Lewis Carroll also belong back with them, though coming on the scene only after the great event' (*Quine 1995*, 254).

Among the evaluative positions, Jean van Heijenoort's distinction between 'logic as calculus' and 'logic as language' (*1967b*) is outstanding. He derived it inductively from an historical comparison of 19th and early 20th century logical systems, thereby creating what can be called the received view (among non-historians) of the history of logic. This distinction was not meant to be purely descriptive: it was also used as a tool for evaluating logical systems, which could be sorted according to the criteria listed. Furthermore, it became a systematically interesting means for understanding formal languages, especially after having been generalized with Jaakko Hintikka's opposition of 'language as calculus' and 'language as a universal medium' (cf. *Hintikka 1988*, *1997a*, *1997c*). This paper will reconsider Jean van Heijenoort's distinction, checking it for historical soundness. It will, however, focus only on two special points, his interpretation of the logical systems in the algebraical and Fregean tradition as related to the older distinction between *calculus ratiocinator* and *lingua characterica*, and the claim that only the Fregean logic contains the essentials of modern quantification theory.

2 Jean van Heijenoort and the Historiography of Logic

2.1 *From Frege to Gödel*

1967 was one of the most important years for the historiography of modern logic. In that year Jean van Heijenoort published his collection *From Frege to Gödel*, according to its subtitle 'A Source Book in Mathematical Logic, 1879–1931'. Its seminal character is due to the fact that for the first time fundamental texts from the history of mathematical logic were gathered together, most of them being published in English language for the first time.

Its formative influence on the historiography of logic had, however, some negative effects due to the one-sided selection of texts.[2] In concentrating on the logical tradition starting with Frege's *Begriffsschrift* (*Frege 1879*) it ignored the logical positions with which the 19th century rebirth of interest in formal logic started: the algebra of logic.[3] In saying that 'the second half of the nineteenth century saw a rebirth of logic' van Heijenoort was clearly aware of the pre-Fregean algebraic tradition of logic (vi). However, he saw this tradition suffering from a number of limitations (*ibid.*):

> It tried to copy mathematics too closely, and often artificially. The multiplicity of interpretations of what became known as Boolean algebra created confusion and for a time was a hindrance rather than an advantage.

No evidence is given for the historical remark. Nevertheless, it led van Heijenoort to his final assessment that the period of the algebra of logic 'would, no doubt, leave its mark upon the history of logic, but it would not count as a great epoch' (*ibid.*). According to van Heijenoort, 'the great epoch in the history of logic did open in 1879, when Gottlob Frege's *Begriffsschrift* was published' (*ibid.*). The reasons were that the *Begriffsschrift* freed logic from the artificial connection with mathematics, at the same time preparing 'a deeper interrelation between these two sciences'. Frege's logic 'presented to the world, in full-fledged form, the propositional calculus and quantification theory' (*ibid.*). It seems to be obvious that this should imply the assertion that there was no quantification theory in the algebra of logic.

2.2 'Logic as Calculus and Logic as Language'

It was in the same year 1967 that van Heijenoort published the paper 'Logic as Calculus and Logic as Language' which was even more important for today's common understanding of the early directions in modern logic. This paper gives an interpretation of Gottlob Frege's discussion of the Boolean logic (esp. *Frege 1882*), which had been provoked by Ernst Schröder's review of Frege's *Begriffsschrift* of 1881 (*Schröder 1881*). Van Heijenoort's discussion is especially based on Frege's assertion that his logic was, unlike Boole's, 'not a *calculus ratiocinator*, or not merely a *calculus ratiocinator*, but a *lingua characterica*' (*van Heijenoort 1967b*, 324, referring to *Frege 1882*, 2). 'If we come to understand what Frege means by this opposition, we shall gain a useful insight into the history of logic' (*van Heijenoort 1967b*, 324).

[2]This was already pointed out in Gregory H. Moore's review of the 2nd ed., *Moore 1977*, 469.

[3]For the mathematical and philosophical contexts of the emergence of the algebra of logic, cf. *Peckhaus 1997*, chs. 5 and 6. Cf. also *Grattan-Guinness 2000* which concentrates on the relation to foundational questions in mathematics.

It would have been useful for reaching this aim if he had brought forward all elements of Frege's assertion. Frege, e. g., also mentioned that he aimed at a 'lingua characterica' according to the Leibnizian model,[4] and he stressed that he still accepted 'this inferring calculation [i. e., the *calculus ratiocinator*] as a necessary component of a concept script' (*Frege 1882*, 2). The distinction between *calculus ratiocinator* and *lingua characterica* is, thus, not a disjunctive one; both aspects occur and have to occur together in a logical system. What Frege wanted to do was to *emphasize* the *lingua characterica* aspect of logic, because his logic, as Frege saw it, differed in purpose from Boole's. Frege did not intend to exhibit an abstract logic in formulae, but wanted to express a content by written signs in a more exact and clearer way than it was possible with words (*Frege 1882*, 1).

This aspect of a union of the distinguished parts is not taken up by van Heijenoort. He interprets the distinction as standing for two kinds of approaches to logic. According to van Heijenoort it is the *lingua characterica* aspect that constitutes the universality of logic. And this universality is regarded as typical for the Fregean approach. It is represented by quantification theory, which provides a vocabulary that the propositional calculus lacks. Whereas in the Boolean propositional calculus the proposition is reduced to a mere truth value, in the Fregean logic,

> with the introduction of predicate letters, variables, and quantifiers, the proposition becomes articulated and can express meaning. The new notation allows the symbolic rewriting of whole tracts of scientific knowledge, perhaps all of it, a task that is altogether beyond the reach of the propositional calculus. We now have a *lingua* not simply a calculus.[5]

Van Heijenoort lists two further consequences of the lingua–calculus distinction and the universality of Fregean logic. Whereas Boole's universal class or De Morgan's universe of discourse can be changed at will, Frege's quantifiers binding individual variables range over all objects. There is no change of universes: 'Frege's universe consists of all that there is, and it is fixed' (*ibid.*, 325). Furthermore, Frege's system is closed, nothing can be outside the system. There are no metalogical questions and no separate semantics.

[4] There is no 'lingua characterica' in Leibniz's works. Leibniz spoke of 'lingua generalis', 'lingua universalis', 'lingua rationalis', 'lingua philosophica', the terms all meaning basically the same. He also introduced the terms 'characteristica' viz. 'characteristica universalis' representing his general theory of signs. Frege obviously took the term 'lingua characterica' from Trendelenburg who used the expression 'lingua characterica universalis' (*1857*, reprint 1867, 6). Cf. *Patzig 1976*, 10, n. 8; *Peckhaus 1997*, 178–181; on Trendelenburg's influence in the history of logic cf. *ibid.*, ch. 4, *Vilkko 2002*, ch. 4.

[5] *Van Heijenoort 1967b*, 325.

'The universal formal language supplants the natural language', he writes (327),

> and to preserve, outside the system, a notion of validity based on intuitive set theory, does not seem to fit into the scientific reconstruction of the language.

The algebraic propositional calculus uses, on the contrary, a model-theoretic approach. Algebraic systems always need interpretations of operation signs and categories, i. e., classes of concepts. They can clearly be divided into a structural, or syntactic, and a semantical side, the latter providing the interpretations of the figures used.

The main features of modern logic as represented by the Fregean system are thus

- quantification theory,
- universality,
- internal semantics,
- fixed universe.

From this it follows that for van Heijenoort the algebra of logic had obviously the following features:

- no quantification theory,
- no universality,
- external semantics,
- no fixed universe.

Van Heijenoort arrives at these results by going back to Frege's own evaluation of his system in comparison with Boole's. Boole is thus taken as the only representative of the algebra of logic. This is justifiable for Frege, at least in the specific context in which he wrote his essay, but strange for van Heijenoort who considers the algebra of logic as if there were no further developments after Boole's death. Van Heijenoort simply ignored the extensions and revisions of the Boolean calculus due to Charles S. Peirce and his school, and to the German algebraist Ernst Schröder, whose work had, unlike Frege's logic, a decisive influence on early 20th century developments (cf., e. g., *Moore 1977*, 469). These shortcomings have clearly been recognized by van Heijenoort's critics.[6]

[6]Van Heijenoort's doctoral student and biographer Irving H. Anellis wrote, e. g., that the 'distinction between the 'algebraic' and the 'quantification-theoretic' traditions is artificial at best' (*1994*, 111). For a survey of criticisms of van Heijenoort as a historian cf. *Béziau 1998/2000*.

3 *Calculus ratiocinator* and *Characteristica universalis*

Before discussing the dispute between Frege and Schröder, Leibniz's distinction between rational calculus and universal characteristics should be considered. It can be found at a great number of places. A typical piece is the short tract 'Fundamenta calculi ratiocinatoris' written presumably during his stay in Vienna between May 1688 and February 1689.[7] In this text Leibniz stresses that all human reasoning is based on the use of signs or characters. Characters are signs perceptible with the senses, e. g. by writing them down, or cutting them into stone. But such abbreviating signs should not only be applied to the things themselves, but also to the ideas of things. 'Abbreviating' means that as soon as a characteristic sign has been established for a complex object, memory can be relieved of the burden of retaining all the characteristic elements of this object. Natural languages are not sufficient for this job of designating objects unambiguously. Only in the language of arithmetic and algebra has this idea been partially realized. All reasoning in these branches consists in using characters. Errors in reasoning prove to be miscalculations. Leibniz was convinced that all human thoughts could be reduced to a few, so to speak primitive, thoughts. Now, if it were possible to relate these primitive thoughts unambiguously to characters, everyone who used these characters in reasoning and writing would either never err, or he would himself recognize his errors with the help of most simple checks. For Leibniz the *ars characteristica* is therefore the true organon or means of a general science which encloses all of man's reasoning. He admits that hitherto no one succeeded in determining how to form such signs. But for now he wants only to show how the order of science would be possible using the general characteristics. He, thus, attempts at a feasibility study. Therefore he uses arbitrarily chosen letters according to the model of mathematics. This notation allows 'calculating with concepts' according to sets of rules, each of them forming a *calculus ratiocinator*.

The characteristic features of the Leibnizian distinction between calculus and characteristics can be summarized as follows:

- The *characteristica universalis* presupposes that the complete list of simple thoughts is at hands.
- These simple thoughts will be unambigously designated with signs, or, if intelligible, with characters. This designation program is more easily

[7] Cf. *Schupp 2000*, X. Schupp published a bilingual (Latin–German) edition of relevant manuscripts for the relation between *calculus* and *characteristics*. 'Fundamenta calculi ratiocinatoris' is published in *Leibniz 2000*, 17–27. The decisive edition is *A* (*Leibniz 1999*) VI.4, no. 192.

realizable the smaller the list of simple thoughts is.

- The complete realization of a *characteristica universalis* may be utopian, so it serves as a heuristic (regulative) idea in the initial phase. Partial realizations, however, are found in the symbolic systems of mathematics. They can be applied at least to a part of real world problems.
- These are not restrictions in principle, but practical restrictions due to the limited powers of man.
- The *calculus ratiocinator* serves for mechanically deducing all possible truths from the list of simple thoughts.
- It forms the syntactic part of the *lingua rationalis*. The characteristics is responsible for the semantic part.
- In the intial period of partial realizations of this program the systems of characters allow various interpretations. In the final (utopian) stage, after having reached the complete designation of all possible simple thoughts, the system would be categorical.

4 The Dispute between Frege and Schröder

Going back to the dispute between Schröder and Frege, it is interesting to see that both, Frege and Schröder, put their logical systems into the logical tradition of Leibniz, and both rely heavily on Adolf Trendelenburg's account of Leibniz's theory of signs (*Trendelenburg 1857*, cf. above, n. 4). Schröder remarks in the opening of his early revision of the Boolean logic *Der Operationskreis des Logikkalkuls* that Leibniz's ideal of a calculus had been realized by Boole, without having been sufficiently recognized after 25 years (*Schröder 1877*. Frege goes into more details in his *Begriffsschrift*. He writes (*1879*, V; *Beaney 1997*, 50):

> Leibniz too recognized—perhaps overestimated—the advantages of an appropriate symbolism. His conception of a universal characteristic, a *calculus philosophicus* or *ratiocinator*, was too grandiose for the attempt to realize it to go further than the bare preliminaries.

Frege is right in stressing the utopian character of the Leibnizian idea, but he nevertheless suggests trying to realize it, at least in parts. (*1879*, VI; *Beaney 1997*, 50):

> But even if this great aim cannot be achieved at the first attempt, one need not despair of a slow, step by step approach. If the problem in its full generality appears insoluble, it has to be limited provisionally; it

> can then, perhaps, be dealt with by advancing gradually. Arithmetical, geometrical and chemical symbols can be regarded as realizations of the Leibnizian conception in particular fields. The *Begriffsschrift* offered here adds a new one to these—indeed, the one located in the middle, adjoining all the others.

The fact that both saw themselves in the Leibnizian tradition grew to a controversy, when Schröder reviewed Frege's *Begriffsschrift* in detail, taking the opportunity to advertise his own algebraic logic in the Boolean tradition as the better alternative. Schröder started his review as follows (*Schröder 1880*, 81):

> This really strange publication — obviously the original work of an ambitious thinker of purely scientific direction of thought — follows a tendency which is of course highly sympathetic for the reviewer who also tried his hand at related subjects. For it promises to step closer towards the Leibnizian ideal of a pasigraphy, which is still far from its realization despite of the great importance attached to it by the ingenious philosopher.

'Pasigraphy' means 'general script'.[8] Schröder here takes up a notion from the discussion on universal languages in the Baroque period. Schröder continues (*1880*, 82),

> Frege's 'concept script' promises to much in its title—strictly speaking: that the contents does not at all conform with it. Instead of tending toward the side of a 'general characteristics' it rather tends definitely—maybe unconsciously for the author—toward the side of Leibniz's '*calculus ratiocinator*'; and the little work makes an attempt in this direction that I would call commendable, even if a great deal of what it aims at has already been done by another party, in fact in an undoubtedly more appropriate manner—as I will show.

It is not difficult to guess that this 'other side' refers to the Boole-Schröderian logic.

Frege responded to these reproaches not only in the published talk 'Ueber den Zweck der Begriffsschrift' (*1882*) mentioned above, but also in a paper entitled 'Boole's rechnende Logik und die Begriffsschrift' written about 1880/81. He sent it to several mathematical and philosophical journals, but received only refusals. Also in this paper Frege started the publicizing his system with expressing his great respect for Leibniz. 'Leibniz had scattered such a wealth of germs of thought, that in this respect hardly anyone can match himself against him' (*Frege 1983*, 9). Among the ideas seemingly dead

[8]For Schröder's conception of pasigraphy cf. *Peckhaus 1990/91*.

and buried in the works of Leibniz, but that might presumably rise from the ashes some day, Frege counts the idea of a *lingua characterica* most closely connected to a *calculus ratiocinator*. According to Frege, Leibniz saw the main advantage of a language in which the concept is composed of its parts and not the word of its sounds in the practicability of some sort of calculation. Frege stressed that of all the expectations Leibniz had in this respect, this one can be shared with greatest confidence (*ibid.*, 9–10). In his small work *Begriffsschrift* he had attempted 'a reapproach to the Leibnizian idea of a lingua characterica' (*ibid.*, 11). In this project he dealt with similar subjects as Boole did before him (12), but Boole had only tried to develop a technique which allows logical problems to be solved in a systematic way, similar to algebra which is a technique for the elimination and calculation of unknowns (13). Contrary to this, Frege had the expression of contents in mind: 'The goal of my attempts is a *lingua characterica* first for mathematics, and not a *calculus* restricted to logic' (13). In the beginning, Frege is quite modest in his goals. He is looking for a symbolic language that concerns only mathematics. It should, however, go beyond a simple calculus, i. e., a system of rules stipulating how to go from given propositions to other propositions without changing truth values. Frege calls such a system of rules a purely logical calculus. He regards mathematical operations like addition or division as operations completely reducible to logical operations. Futhermore, logic serves him for constituting the concepts of such a mathematics. He thus becomes the founder of logicism, i. e. the direction in the philosophy of mathematics that is working on the assumption that all concepts of mathematics (i. e. arithmetic and analysis, but not geometry) can be reduced to purely logical concepts. If this program had succeeded, i. e., if all mathematical theorems had really been presented exclusively with the help of logical expressions, an important aspect of Leibniz's *characteristica universalis* would indeed have been achieved: the demand to keep the number of means for expression as small as possible.

Similar to Frege, in his main logical work, the *Vorlesungen über die Algebra der Logik* (vol. 1, *1890*), Schröder identified as the main goal of Leibniz that of finding an adequate and general designation of the nature of concepts, in such a way that the analysis into their elements would be possible. Then they could be treated by calculation (*Schröder 1890*, 95). Schröder correctly saw the significance of the *characteristica universalis* and the 'ideal of a scientific classification and systematic designation of everything that can be designated' (*ibid.*). However, he stressed that a realization of this ideal would presuppose the complete knowledge of the fundamental operations and the laws according to which they can be applied. Logic has to prepare the ground (*ibid.*). This remark motivates Schröder's focus on the calculus, i. e. the calculation with concepts, but he integrated his logic into the comprehensive semantics of his ‘absolute algebra’, thus realizing also an important aspect of the *characterica universalis* (cf. *Peckhaus 1997*, 254–283).

It has to be noted that both, Frege and Schröder, criticized the concurring system as being a mere *calculus ratiocinator*. Both claimed, on the other hand, that their own system was the better realization of the Leibnizian idea of a *characteristica universalis*. Both accept that a language would require both elements, and both aim at such a language. Schröder’s algebraic attitude requiring an external semantics seems to be closer to the original Leibnizian idea of a *lingua rationalis*. Furthermore, the universality of a universal characteristics is not bound to modern quantification theory.

5 Algebra of Logic and Quantification Theory

In his presentation van Heijenoort heavily relied on Frege’s argument against Boolean logic. Frege indeed (correctly) stressed that his idea of quantification was the decisive mark which distinguished his logic from Boole’s (*1882*, 9). In respect to his universal quantifier he wrote: ‘I regard this way of designation as one of the most important elements of my concept script, by which it gains an important lead over Boole’s writing style, even if it is seen as a pure presentation of the logical forms’ (*ibid.*).

Frege had good reasons for stressing his idea of quantification because Schröder had shown an astonishing lack of understanding in his review. There Schröder had admitted some shortcomings in Boole’s treatment of particular judgements which, addcording to Schröder, had found in Boole’s theory ‘only an inadequate, or, in a rigorous reading no expression’ (*Schröder 1880*, 91). This was harsh, but it was harsh against Boole, not against Frege. Schröder justified his assessment by the following argument (*ibid.*, 90*f.*):

> The indefinite factor v, which is used by Boole to express in the first part of the logical calculus the statement ‘some a are b’ in the form of $va = vb$, does not serve its purpose because this equation is always identically fulfilled by the assumption $v = ab$, even in the case that no a is b. In the section on ‘universality’ *Frege* justly gives such stipulations which allow him to express these judgements indubitably. I will not follow him slavishly in this respect, but rather show that this does not justify his further deviations from *Boole*’s notation, and also that the latter can be modified and extended by analogy. The author reaches this essentially in the way that he introduced Gothic letters in the meaning of general signs and stipulated a notation to negate this universality [. . .].

This is simply not true! Frege’s syntactical sign for universality is the concavity. The Gothic letter signifies the scope of the quantifier, i. e., the range of arguments which can be used in the quantified formula. This quote shows that Schröder (at least in 1880) simply did not grasp the concept of the scope of a quantifier. His easy modification of the Boolean notation consisted in plainly restricting the formula $va = vb$ by introducing the sign $\neq$ for 'not equal' and stipulating that $va \neq 0$ or $ab \neq 0$ which together would also express that some a are b (*ibid.*).

The algebra of logic as of 1880 lacked modern quantification theory (at least Schröder's system did, and those accessible to Schröder at that time). But this state of the algebra of logic should not be taken as essential for the algebra of logic as such. Modern quantification is, on the contrary, an essential part of the algebra of logic, although Frege maintains the priority, of course. Schröder did not introduce modern quantification theory before 1891, when he published his logic of propositions with the second volume of his *Vorlesungen über die Algebra der Logik* (*Schröder 1891*). The step from the calculus of classes (the main topic of the first volume) to the calculus of propositions is taken with the help of an alteration of the basic interpretation of the formulas used. Whereas the calculus of classes was bound to a spatial interpretation especially in terms of the part–whole relation, Schröder now employed a temporal interpretation. He took this idea from Boole's *Laws of Thought* (*1854*, 164–165). This may be illustrated by considering subsumption as the basic connecting relation. In the calculus of classes the subsumption of a to b means that the class a is part of or equal to the class b. In the calculus of propositions this formula may be interpreted in the following way (*Schröder 1891*, § 28, p. 13):

> The time during which a is true is completely contained in the time during which b is true, i. e., *whenever* [. . .] *a is valid*, *b is valid* as well. In short, we will often say: '*If* a is valid, then b is valid,' '*a entails b*' [. . .], 'from a follows b.'

Given the temporal interpretation one may claim, as Ivor Grattan-Guinness does,[9] that quantification theory continued with part-whole theory, but the quotation shows that this is only one possible interpretation. Algebraic quantification theory clearly goes beyond part-whole theory.

Schröder introduces two new logical symbols, the 'sign of products' $\prod$, and the 'sign of sums' $\sum$. He uses $\prod_x$ to express that propositions referring to a domain x are valid for any domain x in the basic manifold 1, and $\sum_x$ to say that the proposition is not necessarily valid for all, but at least for a

[9]Cf. *Grattan-Guinness 2000*, 5. Warren Goldfarb's remark 'There is in the algebra of logic a constant confusion between the class-theoretic and propositional interpretations with mere weight in the end on the former' (*Goldfarb 1979*, 354) leaves open whether it really concerns quantification theory.

certain domain x, or for several certain domains x of our manifold 1, i. e., for at least one x (*Schröder 1891*, § 29, 26–27).

For Schröder the use of $\sum$ and $\prod$ in logic is perfectly analogous to arithmetic. Existential quantifier and universal quantifier are therefore interpreted as possibly indefinite logical addition or disjunction and logical multiplication or conjunction, respectively. This is expressed by the following definition which also shows the duality of $\sum$ and $\prod$ (*Schröder 1891*, § 30, 35).

$$\sum_{\lambda=1}^{\lambda=n} a_\lambda = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n \;\Big|\; \prod_{\lambda=1}^{\lambda=n} a_\lambda = a_1 a_2 a_3 \cdot \ \ldots \ \cdot a_{n-1} a_n \,.$$

Proceeding from this interpretation Schröder can be counted among the precursors of infinitary logic (cf. *Moore 1997*).

It is, thus, obvious that Schröder had a real quantification theory, similar to that of Frege, but small wonder, since Schröder's book was published in 1891, Frege's *Begriffsschrift* in 1879. So Schröder had 14 years to make himself familiar with modern quantification and to adopt it in his own theory. This is a simple and plausible answer, but it is false.

Schröder never claimed any priority for his quantification theory, but he did not take it from Frege. Schröder himself gives the credit for his use of $\sum$ and $\prod$ to Charles S. Peirce and Peirce's student Oscar Howard Mitchell (*Schröder 1891*, 27). About these contributions, Geraldine Brady recently wrote (*Brady 2000*, 6, cf. *ibid.*, 86–94) that in 1883

> one of Peirce's students, O. H. Mitchell, developed a rudimentary system for quantification, limited to a theory of quantified propositional functions with two prenex quantifiers [*Mitchell 1883*]. In the same year, inspired by Mitchell, Peirce introduced quantifiers as operations on propositional functions over a specific domain and part of the semantics of first-order logic for prenex formulas over this domain [*Peirce 1883*]. This direction of research culminated two years later in Peirce's system of first-order logic, which is expressively equivalent to our modern-day first-order logic with functions.

These contributions are later than Frege's, but they are independent from them, born in developing the Boolean kind of logic.

Therefore, quantification theory evidently cannot be the criterion for distinguishing the two big traditions in the history of logic!

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